

PISA 2021 Mathematics: A Broadened Perspective

1. The PISA 2021 Mathematics Strategic Advisory Group was established in March 2017 to provide overall direction as an input to subsequent framework development. The group's final report, below, proposes that the PISA Mathematics framework should be significantly updated, through the introduction of six underpinning mathematical concepts, four new content areas and a number of relevant 21st century skills. It ends with a set of design principles to guide framework and item construction. The report was discussed and supported by the PISA Strategic Development Group (SDG) in October 2017.
2. This document was presented to the PGB at its 44th meeting.
3. The Group's members are as follows:

Name	Country	Title	Field
SCHMIDT Bill (Chair)	USA	University Distinguished Professor, Michigan State University	Statistics
GOOS Marilyn	Australia	Professor of Education, University of Queensland	Algebra
WOLFE Richard	Canada	Professor Emeritus, OISE/U. Toronto	Statistics
JUKK Hannes	Estonia	Lecturer, Institute of Maths and Statistics, University of Tartu	Maths education
GUAN Tay Eng	Singapore	Associate Professor and Head, Maths and Maths education, National Institute of Education	Discrete maths
MARCIAK Zbigniew	Poland	Professor, Mathematical Institute, University of Warsaw	Foundational maths
FERRINI-MUNDY Joan	USA	Deputy Director, National Science Foundation	Calculus
McCALLUM William	USA	University Distinguished Professor of Mathematics, University of Arizona	Geometry

4. Charles Fadel (Founder, Center for Curriculum Redesign and Chair, OECD BIAC education group) has been retained to advise the Group on the role of mathematics in emerging industries and sectors.
5. For most countries mathematical competencies are an expected outcome of schooling. This has been true for a long time. Mathematical competencies initially encompassed basic arithmetic skills, including adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions; computing percentages; and computing the area and volume of simple geometric shapes. In recent times, the digitisation of many aspects of life, the ubiquity of data for making personal decisions involving health and investments, as well as major societal decisions to address areas such as climate change, taxation, governmental debt, population growth, spread of pandemic diseases and the global economy, have reshaped what it means to be mathematically competent and prepared to be a thoughtful, engaged, and reflective citizen.
6. These critical issues as well as others that are facing societies throughout the world all have a quantitative component to them. Understanding them, as well as addressing them, at least in part, requires thinking mathematically. Such thinking is not driven by the basic computational procedures described above, but by mathematical and statistical reasoning, and it demands a reconsideration of what it means for all

students to be competent in mathematics. Mathematical competencies become the new mathematical literacy. They go beyond problem solving, to a deeper level, that of mathematical reasoning, which provides the intellectual acumen behind problem solving.

7. Today's countries face new challenges in all areas of life, stemming from the rapid deployment of computers and robots. For example, the vast majority of students comprising the university freshman class of fall 2017 have always considered phones to be mobile hand-held devices capable of sharing voice, texts, and images and accessing the internet—capabilities seen as science fiction by many of their parents and certainly by all of their grandparents (Beloit College, 2017). The recognition of the growing contextual discontinuity between the last century and the future has prompted a discussion around the development of 21st century skills in students (Ananiadou & Claro, 2009; Fadel, Bialik & Trilling, 2015; NRC, 2012; Reimers & Chung, 2016).

8. It is this discontinuity that drives the need for education reform and the challenge of achieving it. Periodically, educators, policy makers, and other stakeholders revisit public education standards and policies. In the course of these deliberations new or revised responses to two general questions are generated: 1) What do students need to learn, and 2) Which students need to learn what? The most used argument in defence of common mathematics education for all students is its usefulness in various practical situations. However, this argument alone gets weaker with time – a lot of simple activities have been automated. Not so long ago waiters in restaurants would multiply and add on paper to calculate the price to be paid. Today they just press buttons on hand-held devices. Not so long ago we were using printed timetables to plan travel – it required a good understanding of the time axis and inequalities. Today we just make a direct internet inquiry.

9. As to the question of “what to teach”, many misunderstandings arise from the way mathematics is conceived. Some see mathematics as no more than a useful toolbox. A clear trace of this approach can be found in school curricula in many countries. These are often confined to a list of mathematics topics or procedures, with students asked to practice a selected few, in predictable situations. This perspective on mathematics is far too narrow for today’s world. It overlooks key features of mathematics that are growing in importance.

10. Ultimately the answer to the two questions is that every student should learn (and be given the opportunity to learn) to think mathematically, using mathematical and statistical reasoning in conjunction with a small set of fundamental mathematics concepts which themselves are not taught explicitly but are made manifest and reinforced throughout a student’s mathematics learning experiences. This equips students with a conceptual framework by which to address the quantitative dimensions of life in the 21st century. This is the new mathematical literacy which includes problem solving but goes beyond it to becoming mathematically competent.

Defining Mathematical Literacy

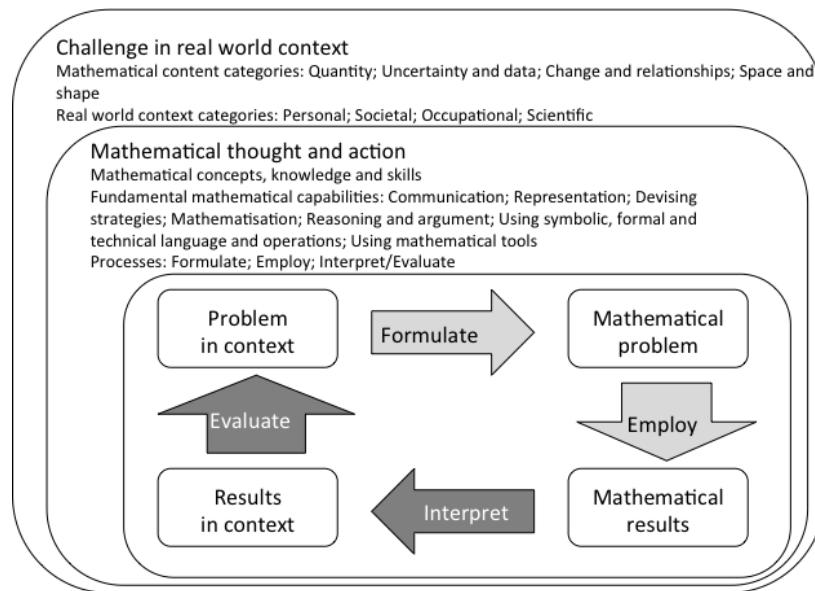
11. The PISA 2012 definition of mathematical literacy was as follows: *An individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make*

the well-founded judgments and decisions needed by constructive, engaged and reflective citizens (p 25, OECD, 2013a).

12. The Assessment and Analytical Framework document clarified the definition by suggesting it reflects “A view of students as active problem solvers”. The document further stated that “The focus of the language in the definition of mathematical literacy is on active engagement in mathematics, and is intended to encompass reasoning mathematically and using mathematical concepts, procedures, facts and tools in describing, explaining and predicting phenomena”. It is important to note that the definition not only focuses on the use of mathematics to solve real-world problems, but also identifies mathematical reasoning as a central aspect of mathematical literacy.

13. The definition is represented pictorially in Figure 1. Further elaborated in the model are the categories of mathematics content knowledge which students must draw on both to formulate the problem by transforming the real world situation into mathematical terms but also to solve the mathematics problem once formulated. Those categories of mathematics content include: quantity, uncertainty and data, change and relationships, and space and shape. Also specified in Figure 1 are the three contexts PISA uses to define real-world situations: personal, societal and scientific.

Figure 1. A Model of Mathematical Literacy in Practice



14. Given this definition, in order for students to be mathematically literate they must be able first to use their mathematics content knowledge to recognise the mathematical nature of a situation (problem) encountered in the real world and then to formulate it in mathematical terms. *This transformation – from an ambiguous, messy, real-world situation to a well-defined mathematics problem – is, perhaps, the critical component of what it means to be mathematically literate. In pursuit of the “broadened perspective” of mathematical literacy referred to in its title, this paper focuses on the delineation of the mathematical competencies needed in this transformation process.* Once the transformation is successfully made, the resulting mathematics problem merely needs to be solved using the mathematics concepts, algorithms and procedures taught in schools. The final component in the PISA definition requires the student to evaluate

the mathematics solution by interpreting the results within the original real-world situation.

15. It is mathematical reasoning that provides the competencies needed to transform the messy, real-world into the world of mathematics. Mathematical procedures can then be used to solve the problem and arrive at an answer.

Evidence on the Role of Mathematical Reasoning in Problem Solving

16. In 2012, PISA incorporated measures of opportunity to learn (OTL) in the student questionnaire to indicate the extent to which students had studied mathematics. This was labelled formal mathematics OTL. Another set of items had students indicate how often they were exposed to applied real-world problems in their classroom instruction and tests. This was termed applied OTL.

17. Given the applied, real-world orientation of the PISA assessment, it was hypothesised that applied OTL would be related to the PISA mathematics literacy test to a greater extent than formal mathematics OTL. In fact both were statistically significant in relation to the overall PISA score, as well as to the seven sub-scores, in most countries. The surprising result, however, was that formal mathematics OTL demonstrated this statistically significant relationship in *every* country, whereas applied OTL was significant in fewer countries – only 79% of the 62 countries/economies that participated in 2012 (see Table 1). In addition, the effect sizes were larger for formal mathematics OTL than for applied OTL.

Table 1. Percentage of PISA Countries with Statistically Significant Relationships of OTL to PISA Performance

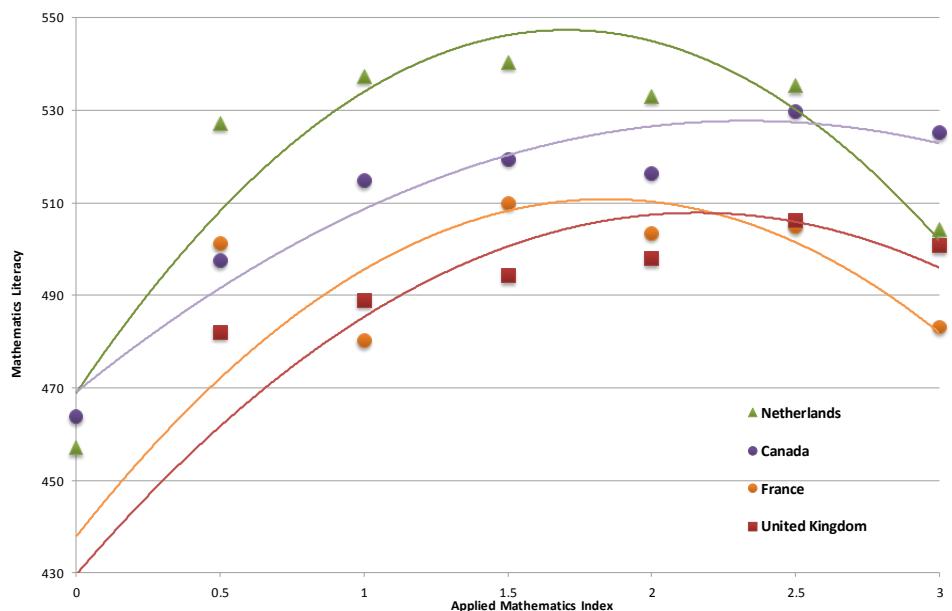
	Effect	Literacy	Change	Quantity	Space	Data	Employ	Formulate	Interpret
Within-School Level	Applied	-2.19*	-2.52	-2.38	-2.02	-2.35	-2.12	-2.45	-2.49
	Math	79%	73%	76%	76%	76%	79%	70%	76%
	Formal	53.15	57.19	52.23	55.87	49.45	51.88	57.37	51.16
	Math	100%	100%	100%	100%	100%	100%	100%	100%
Between-School Level	Applied	-8.54	-8.32	-10.8	-7.67	-8.7	-6.61	-8.98	-10.17
	Math	36%	42%	36%	33%	52%	42%	39%	45%
	Formal	95.16	102.92	94.7	102.21	91.62	96.35	101.01	92.98
	Math	97%	100%	97%	97%	97%	97%	97%	97%
Either Level									
OTL	Applied	79%	73%	76%	79%	82%	85%	76%	82%
	Math	100%							

Note: Average estimated coefficient for those countries with a significant relationship

18. This result prompts the question: why the strikingly strong and consistent relationship of formal mathematics OTL to PISA mathematics literacy? Is it simply due to studying more topics and procedures or is it something else? Again, it is striking that this relationship is even stronger than that demonstrated by applied OTL although this

weaker relationship was still significant in most countries. In addition, applied OTL had a non-linear relationship with PISA mathematics literacy: after a certain point it was negatively related to the PISA score (see Figure 2). The linear relationship for formal mathematics OTL is more straightforwardly interpretable – more OTL is related to a greater literacy score.

Figure 2. The non-linear relationship of Applied Mathematics to performance illustrated for four countries



19. The fact that students with greater formal mathematics OTL, i.e. opportunities to learn advanced mathematics such as complex numbers and trigonometry, do better on the PISA mathematics literacy test supports the hypothesis that mathematical reasoning is a critical component of mathematical problem solving.

20. This hypothesis is especially reasonable given that the PISA 2012 mathematics literacy items do not require knowledge of advanced mathematics such as complex numbers or trigonometry.

21. In other words, it is not the specifics of the advanced mathematics topics studied that are needed to solve the problems on the mathematics literacy test, but rather the increased practice in applying mathematics and reasoning with mathematics.

22. The fact that more mathematics usually means more advanced mathematics, given the hierarchical nature of the discipline, may not be the critical issue. What may well be the critical issue is the greater opportunity to develop the way of thinking logically and reasoning mathematically that is provided by the continuing study of mathematics. This does not imply that this is the only way to acquire such reasoning. *Rather, it is important to understand what it is about mathematics that is central to the development of mathematical reasoning and to incorporate this into the definition of mathematical literacy, into mathematics instruction, and into the PISA assessment.*

The Broader Role of Mathematical Reasoning

23. Mathematical reasoning and solving real-world problems overlap in the core activity of formulating the problem: transforming the messy nature of real-world problems into tightly framed and well-defined mathematics. But there is an aspect to mathematical reasoning which goes beyond solving real-world problems; it is also a way of evaluating and making arguments, interpretations and inferences related to important public policy debates that are, by their quantitative nature, best understood mathematically and statistically.

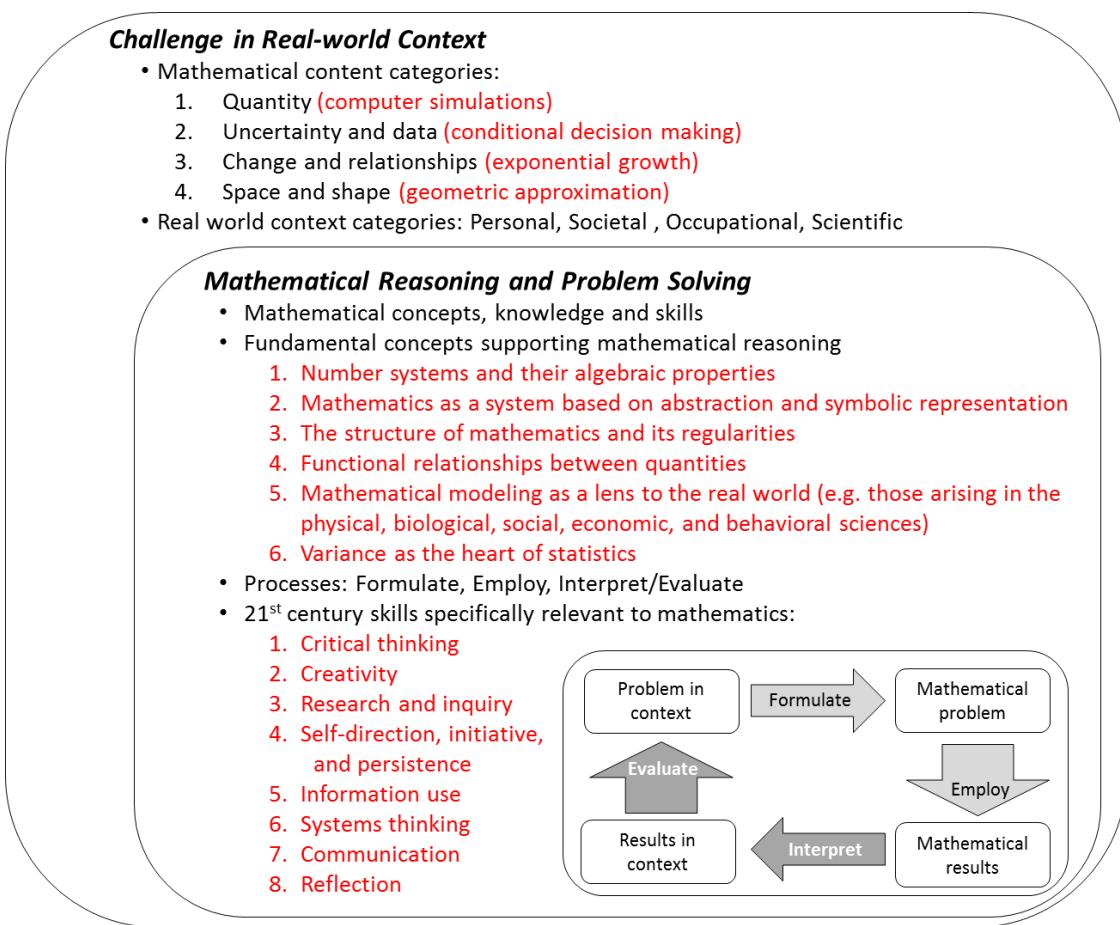
24. We have argued that mathematical literacy comprises two related aspects: *mathematical reasoning*, which plays a key role in being able to apply mathematics to *solve real-world problems*. But mathematical reasoning goes beyond solving problems in the traditional sense of the word to include making informed judgements about those important family or societal issues which can be addressed mathematically. It is here where mathematical reasoning contributes to the development of a select set of 21st century skills.

25. In recognition of the concatenation of the above ideas together with the PISA 2012 model (Figure 1), we have titled this report “PISA 2021 Mathematics: A Broadened Perspective.” We are proposing to broaden the 2012 model in three ways:

- By identifying four areas of emphasis – one under each of the four content categories identified in Figure 1
- By elevating the importance of mathematical reasoning both for the role it plays in problem solving, especially in the formulate stage of the model, and the broader role of being an informed citizen around those important societal issues involving quantitative information. We propose six fundamental concepts that are crosscutting across all of mathematics and provide a foundation for reasoning mathematically.
- By picking out those 21st century skills most closely related to the six fundamental concepts.

26. Taken together, they allow us to go beyond problem solving in measuring mathematical literacy (see Figure 3).

Figure 3. Broadened Model of Mathematical Literacy



Note: The mathematical content category topics listed in parentheses are subtopics from each of the content categories that should receive greater emphasis given their relevance to important societal issues and the nature of the new economy.

27. The three proposals as summarised in Figure 3 show how the 2021 assessment builds directly from the original work done in the PISA 2012 study. Comparing Figure 3 with the original model (Figure 1) shows the increased emphasis on mathematical reasoning so as to deepen the assessment of conceptual understanding as it relates to mathematical literacy. Problem solving retains its place as an important aspect of mathematical literacy but the new model goes beyond it to an even more foundational aspect of mathematical literacy – that of mathematical and statistical reasoning.

28. The detail of the three proposals that follow in the next section should not be taken as un-related to each other. Brought together they create not only a new vision for the 2021 PISA assessment but a way for schooling to support their development. They should neither be taught nor tested separately but in an integrated fashion. They become the three entwined pillars supporting mathematical literacy. Mathematical reasoning (Proposal I) and the six supporting concepts provide the means of addressing problems or broader issues that can be addressed mathematically and increasingly in today's complex world those problems and issues will come from the four mathematics areas listed in Proposal III. Ideally the reasoning and conceptual understanding of the six concepts as

applied in those areas will contribute to the development of a related set of 21st century skills (Proposal II).

Proposal I: Mathematical Reasoning: A Challenging Opportunity for PISA

29. Mathematics is a science about objects and notions which are completely defined, independent of their origin or nature. Once we isolate them in a particular context, they become entities which can be analysed and transformed in ways using ‘mathematical reasoning’ to obtain 100% sure and timeless conclusions. What is also important is that those conclusions are impartial, without any need for validation by some authority. On the other hand, statistics is a science about reasoning with uncertainty or put another way statistics is the search for certainty in the midst of uncertainty.

30. The ability to *reason logically* and to present arguments in honest and convincing ways is a skill which is becoming increasingly important in today’s world. This kind of reasoning is useful far beyond mathematics, but it can be learned and practiced most effectively within mathematics, just because it has the advantage of a fully-defined context, which creates a comfortable training environment and under the assumed axioms the experience of objective truth in a platonic sense.

31. Mathematical reasoning has two aspects, both important in today’s world. One is *deduction from clear assumptions*, which is a characteristic feature of ‘pure’ mathematics. The usefulness of this ability has been stressed above.

32. Another important dimension is *probabilistic reasoning*. At the logical level, there is nowadays constant confusion in the minds of individuals between the *possible* and the *probable*, leading many to fall prey to conspiracy theories or fake news. At the more computational level, today’s world is increasingly complex and its multiple dimensions are represented by terabytes of data. Making sense of these data is one of the biggest challenges that humanity will face in the future. Our students should be familiarised with the nature of such data and making decisions in the context of variation.

33. The power of mathematics, from its very beginnings, lies in the ability of reducing complex contexts to sets of simple basic principles. Euclid’s ‘Elements’ constituted the first spectacular success in this field; he was able to reduce all known ancient geometry to conclusions from five simple assertions. Today’s mathematics theories are no less successful (including the studies on chaos). Good mathematics education should build the attitude for hunting for those ‘prime principles’ in well designed, yet quite complicated contexts.

34. It is our contention that the use of mathematical reasoning, supported by a small number of key concepts that undergird the specific content, skills, and algorithms of school mathematics but also provide a structure in which those specifics are best understood, is the core of mathematical literacy. It is these fundamental concepts that provide the structure and support for mathematical and statistical reasoning. The six fundamental concepts are as follows:

1. Number systems and their algebraic properties

35. Counting is one of the most basic and the oldest of human abstract activities. The basic notion of *Quantity* may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, the world (OECD, 2015, p. 18). At the most

basic level it deals with the useful ability to compare cardinalities of sets of objects. The ability to count usually involves rather small sets - in most languages, only a small subset of numbers have names. When we assess larger sets, we engage in more complex operations of estimating, rounding and applying orders of magnitude. Counting is very closely related to another fundamental operation of classifying things, where the ordinal aspect of numbers emerges. Quantification of attributes of objects, relationships, situations and entities in the world is one of the most basic ways of conceptualising the surrounding world (OECD, 2015).

36. Beyond counting, number is fundamental to measurement, which some would argue is an essential practice in using mathematics to solve problems about our world. As Lord Kelvin once claimed: “When you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind.” (Fey, *On the Shoulders of Giants*, National Academies, 1989).

37. This fundamental concept includes the basic concept of number, nested number systems (e.g., whole numbers to integers to rationals to reals), the arithmetic of numbers, and the algebraic properties that the systems enjoy. In particular, it is useful to understand how progressively more expansive systems of numbers enable the solution of progressively more complex equations. To use quantification efficiently, one has to be able to apply not just numbers, but the number systems. Numbers themselves are of limited relevance; what makes them into a powerful tool are the operations that we can perform with them. As such, a good understanding of the operations of numbers is the foundation of mathematical reasoning.

38. It is important to understand matters of representation (as symbols involving numerals, as points on a number line, as geometric quantities, and by special symbols such as *pi* and *e*) and how to move between them; the ways in which these representations are affected by number systems; the ways in which algebraic properties of these systems are relevant and matter for operating within the systems; and the significance of the additive and multiplicative identities, associativity, commutativity, and the distributive property of multiplication over addition. Algebraic principles undergird the place value system, allowing for economical expression of numbers and efficient approaches to operations on them. They are also central to number-line based operations with numbers, including work with additive inverses that are central to addition and subtraction of first integers and then reals.

39. The centrality of number as a key concept in all the other mathematical areas under consideration here and to mathematical reasoning itself, is undeniable. Students' grasp of the algebraic principles and properties first experienced through work with numbers is fundamental to their understanding of the concepts of secondary school algebra, along with their ability to become fluent in the manipulations of algebraic expressions necessary for solving equations, setting up models, graphing functions, and coding and making spreadsheet formulas. Algebra provides generalisations of the arithmetic in the number systems. And in today's data-intensive world, facility with interpretation of patterns of numbers, comparison of patterns, and other numerical skills are evolving in importance.

2. Mathematics as a system based on abstraction and symbolic representation

40. The fundamental ideas of mathematics have arisen from human experience in the world, in order to give that experience coherence, order, and predictability. Many

mathematics objects model reality, or at least reflect aspects of reality in some way. However, the essence of abstraction in mathematics is that it is a self-contained system, and mathematics objects derive their meaning from within that system. *Abstraction involves deliberately and selectively attending to structural similarities between mathematics objects, and constructing relationships between those objects based on those similarities.* In school mathematics, abstraction forms relationships between concrete objects, symbolic representations and operations including algorithms and mental models.

41. For example, children begin to develop the concept of “circle” by experiencing specific objects that lead them to an informal understanding of circles as being perfectly round. They might draw circles to represent these objects, noticing similarities between the drawings to generalise about “roundness” even though the circles are of different sizes. “Circle” becomes an abstract mathematics object only when it is defined as the locus of points equidistant from a fixed point.

42. Students use representations – whether symbolic, graphical, numerical, or geometric – to organise and communicate their mathematical thinking. Representations can condense mathematical meanings and processes into efficient algorithms. Representations are also a core element of mathematical modelling, allowing students to abstract a simplified or idealised formulation of a real world problem.

3. The structure of mathematics and its regularities

43. When elementary students see

$$5 + (3 + 8)$$

some see a string of symbols indicating a computation to be performed in a certain order according to the rules of order of operations; others see a number added to the sum of two other numbers. The latter group are seeing structure; and because of that they don’t need to be told about order, since if you want to add a number to a sum you first have to compute the sum.

44. Seeing structure continues to be important as students move to higher grades. A student who sees

$$f(x) = 5 + (x - 3)^2$$

as saying that $f(x)$ is the sum of 5 and a square which is zero when $x = 3$ understands that the minimum of f is 5.

45. Structure is intimately related to symbolic representation. The use of symbols is powerful, but only if they retain meaning for the symboliser, rather than becoming meaningless objects to be rearranged on a page. Seeing structure is a way of finding and remembering the meaning of an abstract representation. Being able to see structure is an important conceptual aid to purely procedural knowledge.

46. What is the relationship between mathematical structure and reasoning? As the examples above illustrate, seeing structure in abstract mathematical objects is a way of replacing parsing rules, which can be performed by a computer, with conceptual images of those objects that make their properties clear. An object held in the mind in such a way is subject to reasoning at a higher level than pure symbolic manipulation.

47. A robust sense of mathematical structure also supports modelling. When the objects under study are not abstract mathematical objects, but rather objects from the real world to be modelled by mathematics, then mathematical structure can guide

the modelling. Students can also impose structure on non-mathematical objects in order to make them subject to mathematical analysis. An irregular shape can be approximated by simpler shapes whose area is known. A geometric pattern can be understood by hypothesising translational, rotational, or reflectional symmetry and abstractly extending the pattern into all of space. Statistical analysis is often a matter of imposing a structure on a set of data, for example by assuming it comes from a normal distribution.

4. Functional relationships between quantities

48. Students in elementary school encounter problems where they must find specific quantities. For example, how fast do you have to drive to get from Tucson to Phoenix, a distance of 180 km, in 1 hour and 40 minutes? Such problems have a specific answer: to drive 180 km in 1 hour and 40 minutes you must drive at 108 km per hour.

49. At some point students start to consider situations where quantities are variable, that is, where they can take on a range of values. For example, what is the relation between the distance driven, d , in miles, and time spent driving, t , in hours, if you drive at a constant speed of 108 km per hour? Such questions are the beginnings of thinking about functional relationships. In this case the relationship, expressed by the equation $d = 108t$, is a proportional relationship, the fundamental example and perhaps the most important for general knowledge.

50. Relationships between quantities can be expressed with equations, graphs, tables, or verbal descriptions. An important step in learning is to extract from these the notion of a function itself, as an abstract object of which these are representations. The essential elements of the concept are a domain, from which inputs are selected, a codomain, in which outputs lie, and a process for producing outputs from inputs.

51. Explicitly noting the domain and codomain allows for many different topics to be brought under the function concept. A parametric curve is a function whose domain is a subset of the real numbers and whose codomain is two- or three-dimensional space. Arithmetic operations can be viewed as functions whose domain is the set of ordered pairs of numbers. Geometric formulas for circumference, area, surface area, and volume can be viewed as functions whose domain is the set of geometric objects. Geometric transformations, such as translations, rotations, reflections, and dilations, can be viewed as functions from space to itself.

52. The more formal definition of a function as a set of ordered pairs is both problematic and useful in school mathematics. It is problematic because it removes the dynamic aspect of students' conceptualisation of function: function as process, or mapping, or coordination of two varying quantities. These conceptualisations are useful in many common uses of functions in science, society, and everyday life. On the other hand, the ordered-pair definition emphasises the invariance of the function as an object in its own right, independent of different methods of computing its outputs from its inputs. Thus different forms for the expression of a quadratic function, say

$$f(x) = (x - 1)(x - 3) \text{ and } f(x) = (x - 2)^2 - 1,$$

throw light on different properties of the same object: the one shows its zeros, the other its minimum value.

53. The two views of function—the naïve view as a process and the more abstract view as an object—can be reconciled in the graph of the function. As a set of ordered pairs it is a manifestation of the object. But reading a graph, coordinating the values on

the axes, also has a dynamic or process aspect. And the graph of a function is an important tool for exploring the notion of a rate of change. The graph provides a visual tool for understanding a function as a relationship between covarying quantities.

5. Mathematical modelling as a lens onto the real world (e.g. those arising in the physical, biological, social, economic, and behavioural sciences)

54. Models represent an ideal conceptualisation of a scientific phenomenon. They are in that sense abstractions of reality. A model may present a conceptualisation that is understood to be an approximation or working hypothesis concerning the object phenomenon or it may be an intentional simplification. Mathematical models are formulated in mathematical language and use a wide variety of mathematical tools and results (e.g., from arithmetic, algebra, geometry, etc.). As such, they are used as ways of precisely defining the conceptualisation or theory of a phenomenon, for analysing and evaluating data (does the model fit the data?), and for making predictions. Models can be operated—that is, made to run over time or with varying inputs, thus producing a simulation. When this is done, it can be possible to make predictions, study consequences, and evaluate the adequacy and accuracy of the models.

6. Variance as the heart of statistics

55. One of the aesthetically rewarding aspects of our world is its variability. Living things as well as non-living things vary with respect to many characteristics. However, as a result of that typically large variation, it is difficult to make generalisations in such a world without characterising in some way to what extent that generalisation holds. In statistics accounting for variability is one, if not the central, defining element around which the discipline is based. In today's world people often deal with these types of situations by merely ignoring the variation and as a result suggesting sweeping generalisations which are often misleading, if not wrong, and as a result very dangerous. Bias in the social science sense is usually created by not accounting for the variability in the trait under discussion.

56. Statistics is essentially about accounting for or modelling variability as measured by the variance or in the case of multiple variables the covariance matrix. This provides a probabilistic environment in which to understand various phenomena as well as to make critical decisions. Statistics is in many ways a search for patterns in a highly variable context: trying to find the signal defining “truth” in the midst of a great deal of random noise. “Truth” is set in quotes as it is not the Platonic truth that mathematics can deliver but an estimate of truth set in a probabilistic context, accompanied by an estimate of the error contained in the process. Ultimately, the decision maker is left with the dilemma of never knowing for certain what the truth is. The estimate in the end is a set of plausible values.

Proposal II: Application of Mathematical Reasoning in the Real-World: How 21st Century Skills Fit into PISA 2021

57. There is increased interest worldwide in what are called 21st century skills and their possible inclusion in educational systems. The OECD itself has put out a publication focusing on such skills and has sponsored a research project entitled The Future of Education and Skills: An OECD 2030 Framework in which some 25 countries are involved in a cross-national study of curriculum including the incorporation

of such skills. The project has as its central focus what the curriculum might look like in the future, focusing initially on mathematics.

58. Over the past 15 years or so a number of publications have sought to bring clarity to the discussion and consideration of 21st century skills. One of the more recent reports was produced by the National Research Council (2012) of the United States. While most of these publications focus on the question of what schools need to teach students to know and to do, the NRC report makes a connection between the two questions. The report notes that what many are now referring to as 21st century skills are not something new in the learning enterprise. What may well be different is “society’s desire that *all* students attain levels of mastery – across multiple areas of skill and knowledge – that were previously unnecessary for individual success in education and the workplace”.

59. Lists of 21st century skills that students need to be taught have been based, at least to some extent, on reviews of educational and psychological studies around learning. Such skills have been discussed in the literature using terms such as “deeper learning,” “college and career ready,” “higher order thinking skills,” “new basic skills,” or “next generation learning” (NRC, 2012). Consistent with the viewpoint of earlier reviews the NRC emphasised the conception that *21st century skills are not general skills that are simply applied to various tasks in different contexts but rather “dimensions of expertise” that are intertwined with and specific to a particular domain of knowledge*.

60. Consequently, to underscore this view more completely the authors of the report prefer to “use the term ‘competencies’ rather than ‘skills’”. The rationale for this move is made clear in the earlier work of Anadiadou and Claro (2009) in which they refer to a skill as a component of competence. Competence is then defined as “the ability to apply learning outcomes in a defined context” which involves functional knowledge as well as the application of interpersonal, social, and ethical values. Anadiadou and Claro (OECD, 2009) adopt the formal definition of competence developed by Rychen and Salganik that distinguishes competence and skill: “A competence is more than just knowledge or skills. It involves the ability to meet complex demands, by drawing on and mobilising psychosocial resources (including skills and attitudes) in a particular context” (Rychen & Salganik, OECD, 2003).

61. Anadiadou and Claro develop a framework for 21st century skills that has two dimensions: 1) information, i.e. knowing how to acquire, interpret, and apply appropriate information; and 2) communication which includes the ability to assess and navigate the ethical, social, and interpersonal contexts of the workplace, home, and society. A framework developed by the Partnership for 21st Century Learning (2009) has three dimensions, as elaborated and further developed by Fadel, Bialik & Trilling (ref, 2015):

1. learning and innovation skills such as thinking creatively, working collaboratively, and reasoning effectively;
2. information, media, and technology skills which include accessing, using, and managing information using technology; and
3. life and career skills such as flexibility and adaptability, taking initiative, and working effectively with people of diverse backgrounds.

62. Table 2 is adapted from Table 2-2 in the NRC (2012) report. It summarises and categorises the various terms that have been used to refer to 21st century skills according to three dimensions:

1. *cognitive* including knowledge domains and critical thinking,
2. *intrapersonal* including values, ethics, and self-management, and
3. *interpersonal*, considered a cluster of the two competencies teamwork/collaboration and leadership.

Table 2. Terms for 21st century skills

Dimensions of Competencies	Clusters	Labels/terms referring to 21st Century Skills
Cognitive	Cognitive Processes and Strategies	Critical thinking, problem solving, analysis, reasoning/argumentation, interpretation, decision making, adaptive learning, executive function
	Knowledge	Information literacy, information and communications technology literacy, oral and written communication, active listening
	Creativity	Creativity, innovation
Intrapersonal	Intellectual Openness	Flexibility, adaptability, artistic and cultural appreciation, personal and social responsibility, appreciation for diversity, intellectual interest and curiosity
	Work Ethic/ Conscientiousness	Initiative, self-direction, responsibility, perseverance, productivity, grit, self-regulation, ethics, integrity, citizenship, career orientation
	Positive Core Self-Evaluation	Self-monitoring, self-evaluation, self-reinforcement, physical and psychological health
Interpersonal	Teamwork and Collaboration	Communication, collaboration, teamwork, cooperation, coordination, empathy, trust, service orientation, conflict resolution, negotiation
	Leadership	Leadership, responsibility, assertive communication, self-presentation, social influence with others

Note: Adapted from Table 2-2, NRC, 2012

63. Anadiadou and Claro, reporting results from their survey of how countries are including 21st century skills in their curricula, conclude that most countries “integrate the development of 21st century skills and competencies in a cross-curricular way”. Indeed reviews of 21st century skills typically envision embedding these across the academic content of the curriculum but offer little insight into what this might look like in any one area. The NRC report is an exception as it elaborates how the list of 21st century skills they considered may be expressed through the recent Common Core State Standards for English Language Arts, Mathematics, and the Next Generation Science Standards in the US. Table 3 is an adaptation of Figure 5-2 that identifies the overlap seen between learning expectations in the Common Core State Standards for Mathematics and the 21st Century Skills.

Table 3. Overlap Between Learning Expectations in the Common Core State Standards for Mathematics and 21st Century Skills

Areas of Overlap
Constructing and evaluating evidence-based arguments
Non-routine problem solving
Disciplinary discourse
Systems thinking
Critical thinking
Motivation; persistence
Identity
Self-development
Self-regulation, executive functioning
Collaboration/teamwork

Note: Adapted from Figure 5-2, NRC, 2012

64. The consensus of the various reports is that 21st century skills should be incorporated in some fashion into the current curriculum, i.e. mathematics, science, history, physical education, art, language, etc. One proposed method of incorporation envisions the curriculum as a matrix with the rows defined by the subject matters (mathematics, language, etc.) and the columns defined by the 21st century skills. Each of the cells would then be an opportunity for an infusion of the skills into that subject matter.

65. The advisory group felt this approach to be arbitrary and unrealistic. Not all skills would naturally occur in all subject matters. By natural we mean that the skill is already present or imbedded in the nature of the discipline and its canons of inquiry. This would likely not be true for all skills in a single subject matter.

66. On the other hand, a strong case can be made for the infusion of *specific* 21st Century skills into specific disciplines. For example, it will become increasingly important to teach students at school *how to* make reasonable arguments and *be sure* that they are right. The arguments they make should be strong enough to withstand criticism, and yet, whenever possible, avoid referring to authorities (e.g. ‘Google says so’). This is part of the fundamental competence to make independent judgements and take responsibility for them (OECD, 2005). In the social context it is not enough to be right; one must be able and ready to present arguments and to defend them. Learning mathematics, with its 100% clarity of contexts, is a perfect opportunity to practice and develop the ability for this kind of argumentation.

67. Similarly, in the context of the ‘post-factual’ era, it is urgent to equip students with tools that they can use to defend themselves from lies. Quite often some fluency in logical reasoning is sufficient; a lie usually hides some hidden contradiction. The alertness of young minds towards possible contradictions can be developed most easily in good classes of mathematics.

68. Using the logic of finding the union between generic 21st century skills a related but subject-matter specific skill that is a natural part of the instruction related to that subject matter, the advisory group identified eight 21st century skills for inclusion in the mathematics curriculum and, as such, in the PISA 2021 assessment framework. They are:

- Critical thinking
- Creativity
- Research and inquiry
- Self-direction, initiative, and persistence
- Information use
- Systems thinking
- Communication
- Reflection

69. There always needs to be a context for the use of the 21st century skills. For PISA 2021, the context is mathematics. We therefore propose testing the eight 21st century skills through an item format that allows recording the students' use of mathematical reasoning when solving real-world problems. It is important however to ensure that the item format does not become too burdensome; for example, in having to learn how to navigate the computer when responding to the computer-based items. Such a testing design will require a complex system of scoring that reflects information on the process the student used. Additionally it will require a different and more sophisticated approach to item writing. In effect the items and their scoring rubrics must require that the student demonstrate use of and be evaluated by the appropriate 21st century skill in addressing the problem posed.

Proposal III: Mathematics Content Areas for Emphasis in PISA 2021

70. Four areas of mathematics are proposed for special emphasis in the PISA 2021 assessment. The topics are not outside the domains identified in the PISA 2021 framework, i.e., space and shape, changing and relationships, uncertainty and data and quantity, but are sub-areas of these. In the work of Fadel et al. (“[Recommendations for PISA Maths 2021](#)”, (2017)) the topics are represented not only as commonly encountered situations in adult life in general but as the types of mathematics needed in the emerging new areas of the economy such as high-tech manufacturing. The four are:

- Computer simulations
- Exponential growth
- Conditional decision making
- Geometric Approximation

71. What follows is a brief description of each of the four areas together with example assessment items.

Computer Simulations

The use of standard mathematical algorithms, together with the computer, to solve complex quantitative problems

72. Both in mathematics and statistics there are problems that are not so easily addressed because the required mathematics are complex or involve a large number of factors all operating in the same system. Increasingly in today's world such problems are being approached using computer simulations driven by algorithmic mathematics. A good example is the use of such simulations towards helping individuals plan their retirement so as to have enough money on which to live and accomplish their goals. The number of factors to consider is very large. They include income, age of retirement, expected expenses, investment earnings, stock market values, and proposed age at death,

the values of which require a set of assumptions to be made by the individual the computer program. Changing any of those assumptions individually or collectively provides different results which can then be aggregated statistically to provide an overall estimate of how achievable retirement is, given the goals. Users of such a simulation program need to understand at some level how this is done so as to interpret the results as to the implied impact of their assumptions

Exponential Growth

The growth of a system in which the amount being added to is proportional to the amount currently present

73. Understanding the dangers of flu pandemics and bacterial outbreaks, as well as the threat of climate change, demand that people think not only in terms of linear relationships but recognise that such phenomena need to be represented mathematically with non-linear (exponential) models. Linear relationships are common and are easy to recognise and understand but to assume linearity can be dangerous. A good example of linearity and one probably used by everyone is estimating the distance travelled in various amounts of time while traveling at a given speed. Such an application provides a reasonable estimate as long as the speed stays relatively constant. But with flu epidemics, for example, such a linear approach would grossly underestimate the number of people sick in 5 days after the initial outbreak. Here is where a basic understanding of exponential growth and how rapidly infections can spread given that the rate of change increases from day to day is critical. The recent spread of the Zika infection is an important example of exponential growth; recognising it as such helped medical personnel to understand the inherent threat and the need for fast action.

Conditional Decision Making

Decision-making in a probabilistic environment, using all the relevant information

74. Going to the doctor often ends up requiring a decision about what to do next. Do I take the medicine? Given my age, the doctor says that I should live another 20 years. The question arises – should I take a low dosage aspirin every day? This is a decision for which certainty is not possible. However, the decision making can be improved by asking the doctor how long I should live and not have a heart attack if I have a particular blood pressure, heart rate, cholesterol-level and body fat ratio.

75. This is a question answerable best in terms of conditional probability. Such a probability or even an estimate of that probability takes into account the additional information such as the patient's current health as indicated by the blood and other tests. If there is a relationship between blood pressure and cholesterol levels and the other test results in terms of the chances of having a future heart attack, the predicated conditional mean might for example change the life expectancy estimate of 20 years to only 5 years, impacting decisively on the patient's view of whether to take a daily aspirin. Essentially the conditional mean provides a non-technical approach to regression analysis.

76. Variables which are dichotomies or polychotomous in nature must be analysed differently to continuous variables such as blood pressure and age. Questions such as do people with blue eyes also tend to have blonde hair, or how many ways can 20 people be organised into the 11 positions on soccer team are examples of situations where knowing the basic rules of combinatorics would help in making decisions. In the case of the cross-classification of categorical variables, such as blue eyes/or not vs. blonde

hair/or not, understanding the different types of percentages that can be computed and what each of them means is critical to understanding the phenotypic genetics. Combinatorics (Into how many different orders can 5 objects be placed? Into how many combinations can 16 objects be placed, taking 2 at a time?) also help to understand commonly occurring situations. Game theory, a more formal approach, can be applied to decision making between different categorical options such as winning vs. losing, using many of these same approaches together with conditional probability.

Geometric Approximation

Shapes that do not follow typical patterns of evenness or symmetry

77. Today's world is full of shapes that do not follow typical patterns of evenness or symmetry. For example, architectural designers often deploy traditional geometric objects (lines, angles, squares, triangles and circles) in non-traditional ways, in order to enhance the use of space and aesthetic beauty. The concept has entered our lives through the design of our homes, the art we buy, the urban design of our cities and the places in which we work, just to name a few.

78. Because simple formulas do not deal with irregularity, it has become more difficult to understand what we see and find the area or volume of the resulting structures. For example, finding the needed amount of carpeting in a building in which the apartments have acute angles together with narrow curves demands a different approach than would be the case with a typically rectangular room. Finding the shortest distance between two points also becomes less obvious.

Some Directions for the Contractors

79. In this document the Special Advisory Group has laid out a vision of a broadened definition of mathematical literacy. It builds directly on the 2012 framework and retains a central focus on the problem solving model as illustrated in Figure 1. It, however, includes a second central focus, that of mathematical reasoning. These two aspects of mathematical literacy are what need to be built into the PISA 2021 assessment. Mathematical reasoning, the more basic or fundamental of the two as well as the more general, is likely present but not explicit in the problem solving aspect, especially in the first stage of the model – formulating the messy somewhat ill-defined “real-world” problem mathematically.

80. The task of developing a test specific framework or blueprint for the 2021 assessment must not only continue to incorporate problem solving but now must also include the separate aspect of mathematical reasoning, focusing on the six fundamental concepts supporting it. In fleshing this out into the test blueprint, attention should also be given to their relationship to the eight 21st century skills.

81. In effect, the test design has 6 factors (see Figure 4 which is a reformatted version of Figure 3 that facilitates this discussion by more clearly delineating the factors and their levels). Each factor is listed below with the number of aspects (levels) included in parentheses:

- Mathematical reasoning and problem solving foci (2)
- Real-world contexts (4)
- Mathematics content categories and emphases (4)
- Fundamental concepts supporting mathematical reasoning (6)

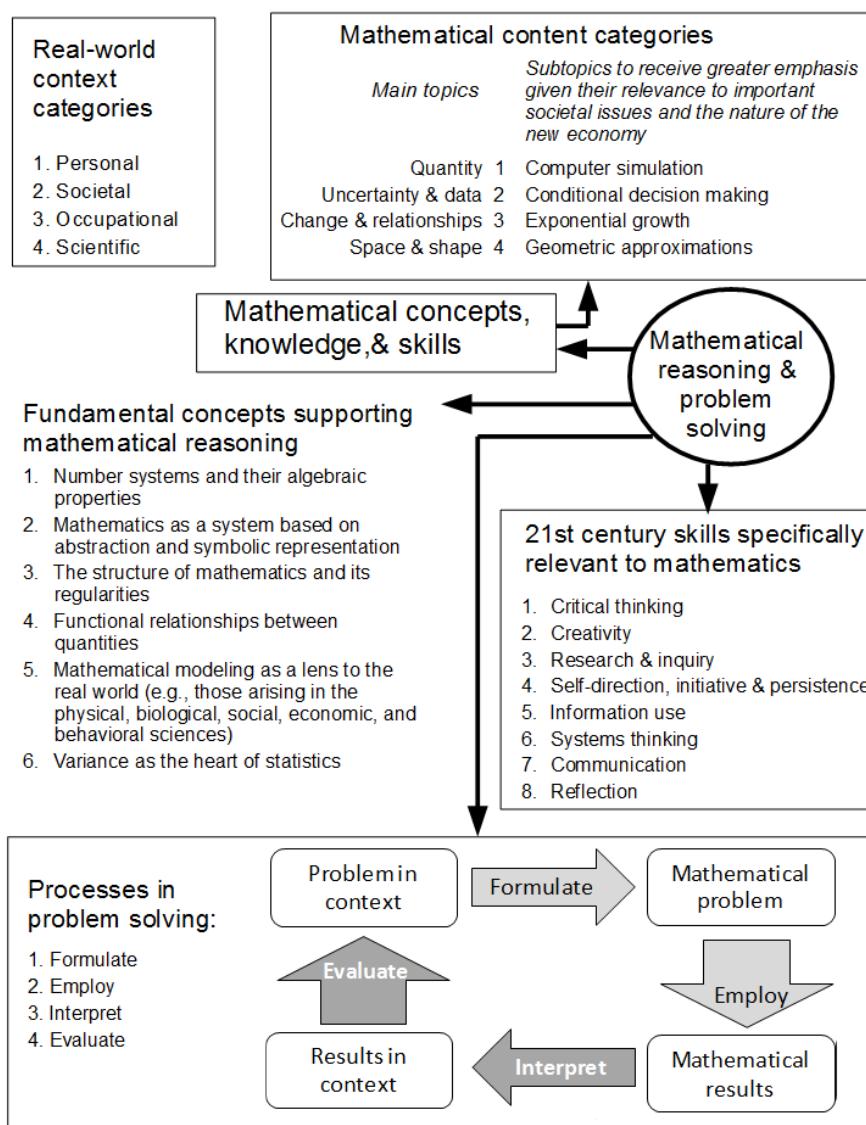
-
- 21st century skills specifically relevant to mathematics (8)
 - Processes in problem solving (4)

82. There is some implicit nesting among these factors: The processes in problem solving are aspects of the main problem solving focus. The fundamental concepts supporting mathematical reasoning are aspects of the mathematical reasoning focus. It is also clear that mathematical reasoning is important in problem solving but that mathematical reasoning is present in situations other than problem solving, such as in evaluating arguments, interpretations, and inferences.

83. This test design is obviously too big as the number of potential cells vastly exceeds the number of items that can be included on a PISA test. The contractor needs to look at ways to address this, for example by building a partial replicate of the full design that limits the number of needed items. This must be examined carefully looking at the alias structure of different partial replicates to determine the necessary assumptions regarding various interaction terms. Important substantive considerations are essential in making those assumptions as well as in making sure the desired subtest scores from a substantive perspective are estimable. Clearly, part of that effort will need to include choosing areas of emphasis and collapsing across aspects within a factor as well.

Figure 4. The Broadened Model of Mathematical Literacy (expanded)

Reformatted version of Figure 3

Challenge for Mathematical Literacy in Real-World Context

84. We respect the professionalism of the contractor and feel the task of working this out as well as other technical issues is their responsibility and not ours. But we also feel that there are special concerns we have as mathematicians, statisticians and mathematics educators that derive from the content itself and are critical to the success of measuring mathematical reasoning within and outside of how it interacts with problem solving. Mathematical reasoning may be more fundamental as we have argued but it will also be more difficult to measure in terms of the response data needed and also as to how we map those data into meaningful score points.

85. With that in mind, we offer several special content relevant concerns, issues and suggested approaches in the form of guidelines for the contractor to take into consideration:

- In the final analysis the test design must represent mathematical literacy including both aspects – mathematical reasoning and problem solving – as articulated in and contextualised in Figures 3 and 4. In other words, the test design must generalise to the population of tasks. Weighting will need to be specified determining the relative contribution of various factors.
- Given that mathematical reasoning is a sub-score of mathematical literacy, as is problem solving, we recommend the distribution of score points be allocated as 60% problem solving and 40% mathematical reasoning (or 67/33) leaving a substantial overlap with 2012. Ultimately consideration of a 50/50 distribution would be warranted but for continuity with the 2012 we recommend the former for 2021.
- Some factors such as “real-world contexts” and “mathematics content categories” could be considered not as creating marginal sub-scores but enter the design randomly in a balanced fashion across the fixed factors for which sub-scores will be developed. Essentially in 2012 that was what was done with the real-world context. The four content areas seem to add little and be mostly redundant with the total score but it is likely that the OECD will want to continue them. We have suggested subtopics that seem to be more interesting and relevant than the general content areas.
- Three mathematical reasoning sub-scores are proposed by the Special Advisory Group including: mathematical modelling (5), the structure of mathematics (3), and variance (6).
- For the 21st century factor, two sub-scores are proposed: communication (6) and persistence (4).
- This creates a mathematical literacy score; a mathematical reasoning score, as well as three sub-scores including the structure of mathematics, variance and mathematical modelling. Also included is a 21st century skills score and two sub-scores representing communication and persistence. In addition there would be a problem solving score and three sub-scores related to the problem solving model: formulate, employ and interpret/evaluate – a total of 12 scores. If the four content category sub-scores are developed as they were in 2012 that would produce 16 scores. Even 12 scores appears to be too large a number.
- New scoring approaches need to be developed around partial credit. The kind of items that need to be developed to measure mathematical reasoning call for open-ended responses more than choosing from multiple options. We propose including partial credit for the correct recognition or setting up of a problem and not just the final correct answer as well as credit for trying to understand an argument or solve a problem even if totally wrong in the approach. Here the mere attempt at a solution would include credit.
- The contractor also needs to be prepared for large amounts of qualitative data resulting from such item types. This is not just for purposes of scoring the items but such data become part of the results that need to be summarised for report writing. This will take a substantial amount of work.
- The scores derived here will not typically be unidimensional. Items may be needed to contribute to multiple sub-scores. Multidimensional scaling should also be considered.

86. All of the above proposed guidelines essentially focus on one issue that the Strategic Advisory Group wants the PISA2021 test framework and corresponding assessment to have – content validity. The measurement validity and relevance of the test and its reporting is what is most important. We recognise that the type of measurement we propose is difficult, in fact *very* difficult. As some of the members of the Group are psychometricians, we know what the arguments against such an approach will be regarding reliability and other technical issues, but the committee feels those arguments need to be challenges, not barriers. What we want to measure is important and when we do it, it will likely have a strong positive impact on mathematics instruction across the world. We know what we want to measure, i.e. mainly the underlying reasoning and problem-solving abilities; what we need are the means by which to do it. This is the challenge that faces PISA and the contractors.

Some Item Types for Consideration

87. The following 10 items are included as examples of the types of items that will be needed. There is no particular order in the presentation of these items. Although not written explicitly for the purposes of this proposal, they represent the range of what needs to be developed.

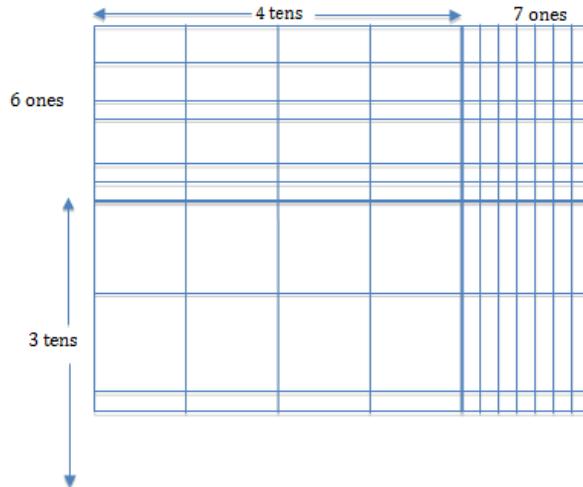
Figure 5. Sample Item 1

A group of children are learning how to multiply 2-digit numbers in different ways.

In the examples below they are calculating 47×36 .

- Explain:
- 1) how the calculation is done in each representation
 - 2) the connections between all three representations

Representation 1



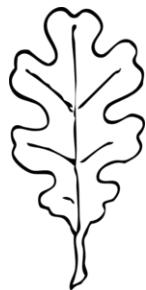
Representation 2

X	40	7	
6	240	42	282
30	1200	210	1410
			1692

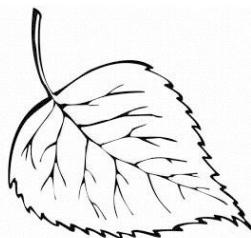
Representation 3

Th = Thousands
 H = Hundreds
 T = Tens
 O = Ones

Th	H	T	O
	X	4	7
		3	6
		4	2
	2	4	
1	2	1	
1	6	9	2

Figure 6. Sample Item 2

OAK



HORNBEAM



MAPLE

Estimating the surface area of tree leaves.

- Above are drawings of leaves from three trees. Which leaf has the greatest surface area? How would you measure the areas to check which is largest? Explain what problems there are in your measurement procedure.
 - These are just particular leaves from the three kinds of trees. The actual leaves on a tree vary somewhat in size from one to the next. Suppose you had a sample of, say, 100 leaves from each tree. How would you estimate the average surface area of leaves of each type.
- Note. It might be tiresome to apply the method you suggested in part a) so many times in part b)!

Figure 7. Sample Item 3

There are three stores in the town of Gluck which sell CDs. The prices vary across the stores ranging from \$3 to \$10 depending on the particular CD. In store A, the prices range from \$4 - \$8. In store B the cost of the same CDs range from \$3 - \$9. The third store sells all its CDs for \$5 -\$7.

- You want to buy several different CDs and are not sure how much the ones you want cost. You only have time to go to one store. Which store would you pick – A, B, or C – and why?
- Assume you just want to buy several CDs and don't really care about specific ones, which store would you pick and why?

Figure 8. Sample Item 4

A fatal hit and run accident occurred on a dark and stormy night in Gotham City. A suspect driving a damaged red truck was arrested. A witness to the accident has given evidence that the hit and run vehicle was a blue truck. The following facts are available to you as the judge.

- i) The witness is truthful.
 - ii) The witness was tested earlier with a blue truck and a red truck in similar weather conditions. She was able to correctly identify a blue truck 8 out of 10 times and correctly identify a red truck 6 out of 10 times.
1. What is the probability that the witness saw a blue truck when the truck was actually red?
2. What is the probability that the witness saw a blue truck when the truck was actually blue?
- The following information is available to you.
- iii) There are 1000 red trucks and 50 blue trucks in Gotham City.
 - iv) 50% of the red trucks and 80% of the blue trucks were on the road at the time of the accident.
 - v) Red trucks are generally more visible than blue trucks.
 - vi) All trucks in Gotham City are either red or blue.
 - vii) The suspect knew the victim.
3. Incorporate the additional information above which you think is relevant. What would you as judge rule concerning the suspect? Explain and support your conclusion.

Figure 9. Sample Item 5

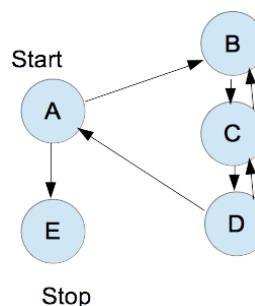
Suppose it is known that over the 100 year history of the town that each day there is a 1 in 5 chance that the brown bears that live in the forest near the city come to the garbage dump where people can observe them. You notice on your cellphone that the city has announced that they were there yesterday so you decide not to go to observe for the next 4 days.

- 1) Is that a wise decision; why is it or isn't it?
- 2) How would you time your trip if you wanted to watch the bears?

Figure 10. Sample Item 6

Shopping at the new store in town includes a 43% discount on all items. The state you live in has a 7% sales tax. You want to buy many things but only have a total of \$52 hours to spend. How can you decide what to buy?

- 1) Describe in words what you could buy.
- 2) Make up an algorithm by which to determine what you can buy.
- 3) Write a decision model to help you.

Figure 11. Sample Item 7

This diagram defines a simple machine for producing strings of letters. You start with position "A" and write down that letter. Then you go to any position connected with an arrow to your current position. Write down the letter you find there. You continue from position to position, but if you reach position "E", you stop.

- a) What is the shortest string you can obtain?
- b) What is the longest string?
- c) Can you describe the set of all strings that can be obtained?

Figure 12. Sample Item 8

- Fine Needle Aspiration Cytology is a biopsy method for testing women for breast cancer. A biopsy is a tissue sample taken from the body so that it can be examined more closely.
- No medical test is 100% accurate – sometimes a positive result is returned for a healthy person (a false positive result) or a negative result for someone who actually has the disease (false negative).
- The incidence of women biopsied who actually have breast cancer is 30%. The FNA biopsy has a 2% probability of giving a false positive result and a 14% probability of giving a false negative result.
- If 100,000 women were biopsied for breast cancer, how many of these women actually have breast cancer? How many would receive a false negative biopsy result? How many would correctly receive a positive biopsy result?
- How many of the 100,000 biopsied women don't have breast cancer? How many of these women would receive a false positive biopsy result?
- What is the probability that a woman with a negative biopsy result actually has breast cancer?

Figure 13. Sample Item 9

Population density varies greatly between countries, and also within countries, because much of the Earth's land surface is not suitable for human habitation. Population density is measured by the number of people divided by land area in square kilometers.

Country	Population (2016)	Land area (km^2)
New Zealand	4.7 million	68,918
Ireland	4.7 million	263,884

New Zealand and Ireland are two countries with the same population but different land areas.

- 1) Calculate the population densities for the two countries. (NZ 17.8 people/ km^2 , Ireland 68.2 people/ km^2).

Because it can be difficult to image what a square kilometer looks like, Liam and Naomi are thinking of different ways to visualise population density.

- 2) Liam imagines that all the people in Ireland join hands and stretch themselves out in a straight line.

- a) How long would this line be? Write out an explanation for this calculation that one of your classmates would understand.
- b) How does the length of this line compare to something in the real world that is familiar to you? [e.g., distance between two towns, length of a football field]

- 3) Naomi imagines that all the people in New Zealand want to stand close together on football fields measuring 100m by 70m.

- a) How many people could stand close together on one football field? Write out an explanation for this calculation that one of your classmates would understand.
- b) How many football fields would it take to hold the entire population of New Zealand?

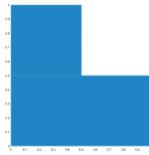
- 4) Which of these two approaches – Liam's or Naomi's – do you think is better for understanding and visualising population density? Write a letter to Liam and Naomi to explain your decision to them.

- 5) Singapore has a population of 5.7 million people and a land area of 719 km^2 . Use your preferred method described in (4) to represent the population density of Singapore and compare this to the population densities of both New Zealand and Ireland.

Figure 14. Sample Item 10

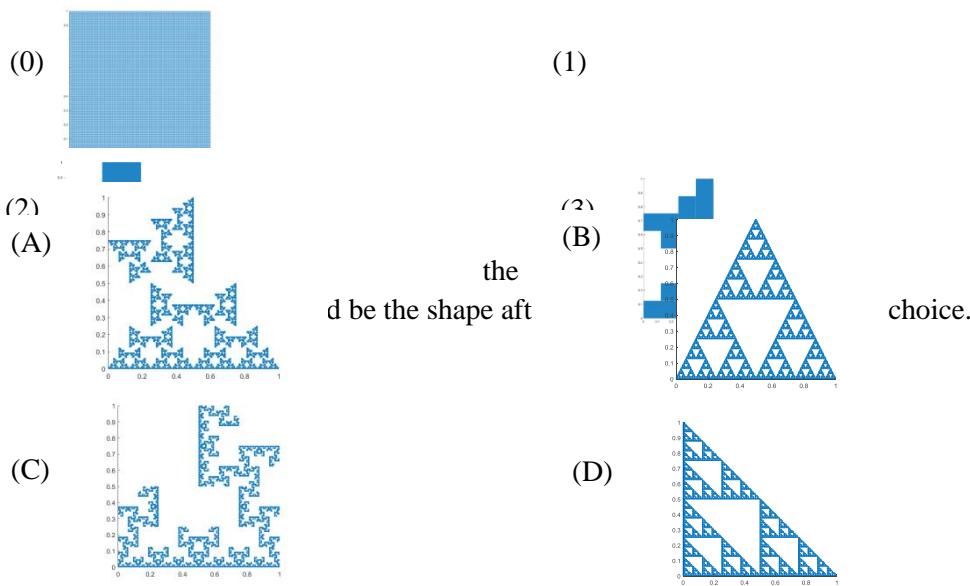
A designer uses a computer to create a tile pattern as follows. A square of side 1 undergoes a transformation f that consists of 3 concurrent sub-transformations:

1. An enlargement of factor 0.5, followed by an anti-clockwise rotation of 90° , followed by a translation 0.5 units to the right and 0.5 units up.
2. An enlargement of factor 0.5, followed by a reflection about the y -axis, followed by a translation 0.5 units to the right.
3. X.



The resulting shape is made to undergo the transformation f . This is repeated 20 times.

The figures below show the original square and the results of the first 3 iterations:



Final Thoughts

88. The three suggestions made for broadening the PISA 2021 Assessment Framework fit well with the original 2012 PISA framework. That was the advisory group's intention. The original 2012 framework focuses on problem solving and outlines the steps involved in that process. Also contained in the 2012 framework are four broad areas of mathematics. Proposal III adds emphasis to certain topics in each of those four original domains of content, reflecting areas that are more commonly found and salient in today's world. Proposal I focuses on identifying six fundamental concepts that transcend and provide structure, to the definition of mathematics reasoning which is proposed as a second aspect of mathematical literacy in addition to problem solving.

89. Finally, Proposal II adds the 21st century skills which are an abstraction and generalisation of many of the mathematics processes used in the solution of problems as defined by the original 2012 model. Students using those skills in solving the problems are using the kinds of mathematical processes that generalise beyond mathematics to life more broadly. This is not artificial as these processes are inherent in and a major part of mathematics. They answer the age old question of, when am I going to use this math in my real life anyway? This is what is meant by quantitative literacy, the ability to apply those mathematical concepts and processes to different real-life situations. Perhaps, even in situations in which they have no idea of the traditional mathematics used to solve it. The question is whether the student can adapt using his/her understanding of the six elements supporting mathematical reasoning in Proposal I to find a solution to the problem. Generalised, to the extent possible, those become 21st century skills. Whether these skills are generalisable is still an open research question while some earlier studies would suggest not. Measuring them, demands that we be able to record the process by which students attempt those solutions and that demands a non-traditional item format that could be nicely supported given that the 2021 test will be computer based.

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