Marginal stability in critical network-economies

Théo Dessertaine, J. Moran, J-P. Bouchaud, M. Benzaquen
Ecole polytechnique & Capital Fund Management

NAEC - X Day at OECD
January 21st, 2019

Context & motivations

The **"Small shocks, large business cycle"** (B.Bernanke, late 2000s chairman of the Fed) puzzle:

- ⋆ Aggregate (as in summed over firms) volatility however remains very high (US GDP growth since 1950 $\sim 3\% \pm 2.5\%$)

A host of explanations has emerged:

- ⋆ Granularity: the fluctuations of entities (firms) with a fat-tailed distrib. do not necessarily average out (X.Gabaix)
- ⋆ **Propagation of shocks along the supply chain network (Acemoglu, Carvalho et al.)**

Can we have a model accounting for a propagation of shocks along the supply chain network?

The **"Small shocks, large business cycle"** (B.Bernanke, late 2000s chairman of the Fed) puzzle:

- ⋆ Aggregate (as in summed over firms) volatility however remains very high (US GDP growth since 1950 $\sim 3\% \pm 2.5\%$)

A host of explanations has emerged:

- ⋆ Granularity: the fluctuations of entities (firms) with a fat-tailed distrib. do not necessarily average out (X.Gabaix)
- ⋆ **Propagation of shocks along the supply chain network (Acemoglu, Carvalho et al.)**

Can we have a model accounting for a propagation of shocks along the supply chain network?

The **"Small shocks, large business cycle"** (B.Bernanke, late 2000s chairman of the Fed) puzzle:

- ⋆ Aggregate (as in summed over firms) volatility however remains very high (US GDP growth since 1950 $\sim 3\% \pm 2.5\%$)

A host of explanations has emerged:

- ⋆ Granularity: the fluctuations of entities (firms) with a fat-tailed distrib. do not necessarily average out (X.Gabaix)
- ⋆ **Propagation of shocks along the supply chain network (Acemoglu, Carvalho et al.)**

Can we have a model accounting for a propagation of shocks along the supply chain network?

The **"Small shocks, large business cycle"** (B.Bernanke, late 2000s chairman of the Fed) puzzle:

- ⋆ Aggregate (as in summed over firms) volatility however remains very high (US GDP growth since 1950 $\sim 3\% \pm 2.5\%$)

A host of explanations has emerged:

- ⋆ Granularity: the fluctuations of entities (firms) with a fat-tailed distrib. do not necessarily average out (X.Gabaix)
- ⋆ **Propagation of shocks along the supply chain network (Acemoglu, Carvalho et al.)**

Can we have a model accounting for a propagation of shocks along the supply chain network?

The **"Small shocks, large business cycle"** (B.Bernanke, late 2000s chairman of the Fed) puzzle:

- ⋆ Aggregate (as in summed over firms) volatility however remains very high (US GDP growth since 1950 $\sim 3\% \pm 2.5\%$)

A host of explanations has emerged:

- ⋆ Granularity: the fluctuations of entities (firms) with a fat-tailed distrib. do not necessarily average out (X.Gabaix)
- ⋆ **Propagation of shocks along the supply chain network (Acemoglu, Carvalho et al.)**

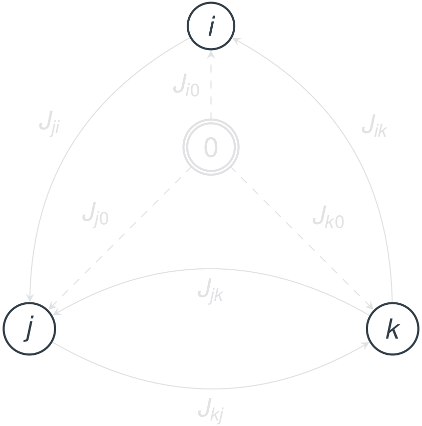
Can we have a model accounting for a propagation of shocks along the supply chain network?

Modelisation

We make a distinction between **two kinds of variables**

FAST	SLOW
⋆ Prices of goods p_i ⋆ Production of firms λ_i ⋆ Consumption of households ξ_i	⋆ Firm's technology z_i ⋆ Supplier-buyer network J_{ij} ⋆ Household preferences θ_i ⋆ Available labour V_i

STRATEGY: study how the fast variables behave depending on the configuration of the slow variables.

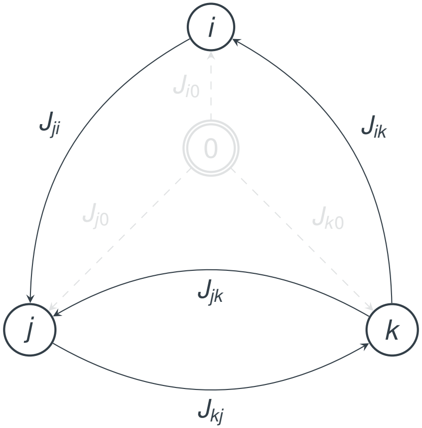


FIRMS

- ⋆ n firms labelled $i, j, k \ldots$ (i)
- ⋆ Input-output network J_{ij}: if j is a supplier of i, $J_{ij} > 0$
- ⋆ Leontieff production function with technology z_i and exchanged quantity Q_{ij}

$$\pi_i = z_i \min_j \left(\frac{Q_{ij}}{J_{ij}} \right)$$

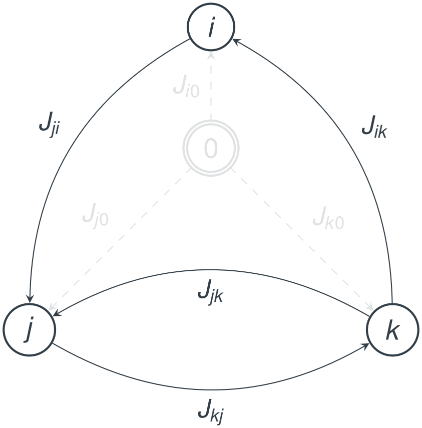
Slow variables: the static framework



FIRMS

- ⋆ n firms labelled $i, j, k \dots$ ⓘ
- ⋆ Input-output network J_{ij}: if j is a supplier of i, $J_{ij} > 0$
- ⋆ Leontieff production function with technology z_i and exchanged quantity Q_{ij}

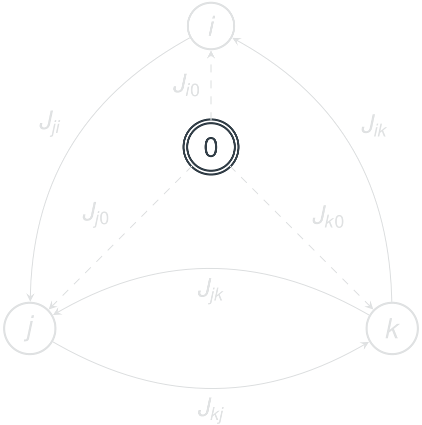
$$\pi_i = z_i \min_j \left(\frac{Q_{ij}}{J_{ij}} \right)$$



FIRMS

- ⋆ n firms labelled $i, j, k \ldots$ ⓘ
- ⋆ Input-output network J_{ij}: if j is a supplier of i, $J_{ij} > 0$
- ⋆ Leontieff production function with technology z_i and exchanged quantity Q_{ij}

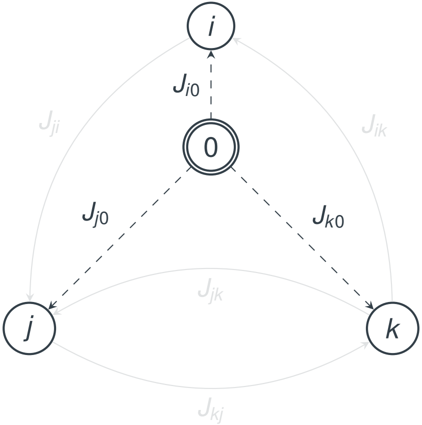
$$\pi_i = z_i \min_j \left(\frac{Q_{ij}}{J_{ij}} \right)$$



HOUSEHOLD

- ⋆ 1 representative household (0)
- ⋆ Work J_{i0} provided by the household to the firms
- ⋆ Consumption ξ_i and preferences θ_i for good i to define log-utility

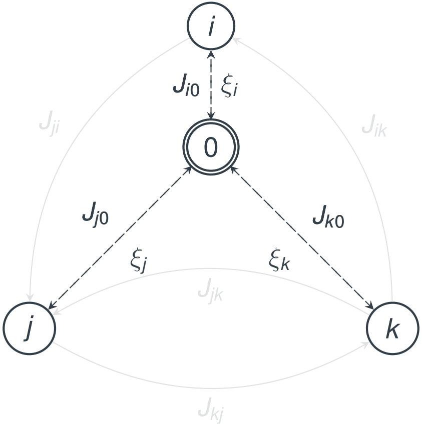
$$\mathcal{U} = \sum_i \theta_i \log \xi_i$$



HOUSEHOLD

- ⋆ 1 representative household (0)
- ⋆ Work J_{i0} provided by the household to the firms
- ⋆ Consumption ξ_i and preferences θ_i for good i to define log-utility

$$\mathcal{U} = \sum_i \theta_i \log \xi_i$$

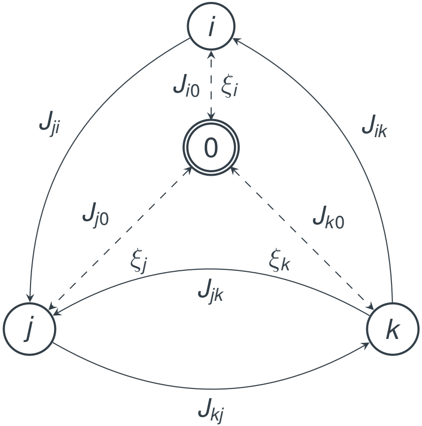


HOUSEHOLD

- ⋆ 1 representative household (0)
- ⋆ Work J_{i0} provided by the household to the firms
- ⋆ Consumption ξ_i and preferences θ_i for good i to define log-utility

$$\mathcal{U} = \sum_i \theta_i \log \xi_i$$

Slow variables: the static framework



Fast variables

In general, we expect some **dynamical evolution** for the fast variables

$$x_i(t+1) = g_i\left(\{x_j\}, t\right)$$

Dynamical evolution of prices $p_i(t)$ and productions $\lambda_i(t)$? Following **rule of thumb**

- ⋆ If a firm is profitable, it will increase its output

$$\lambda_i(t+1) = \lambda_i(t)\left(1 + \beta\frac{P_i(t)}{z_i p_i(t)\lambda_i(t)}\right)$$

  where $P_i(t)$ is the profit of firm i at time t.

- ⋆ If a firm has excess supply, it will decrease its prices

$$p_i(t+1) = p_i(t)\left(1 - \alpha\frac{e_i(t)}{z_i\lambda_i(t)}\right)$$

  where $e_i(t)$ is the production surplus of firm i at time t.

Fast variables

In general, we expect some **dynamical evolution** for the fast variables

$$x_i(t+1) = g_i\left(\{x_j\}, t\right)$$

Dynamical evolution of prices $p_i(t)$ and productions $\lambda_i(t)$? Following **rule of thumb**

- ⋆ If a firm is profitable, it will increase its output

$$\lambda_i(t+1) = \lambda_i(t)\left(1 + \beta\frac{P_i(t)}{z_i p_i(t)\lambda_i(t)}\right)$$

where $P_i(t)$ is the profit of firm i at time t.

- ⋆ If a firm has excess supply, it will decrease its prices

$$p_i(t+1) = p_i(t)\left(1 - \alpha\frac{e_i(t)}{z_i\lambda_i(t)}\right)$$

where $e_i(t)$ is the production surplus of firm i at time t.

Fast variables

In general, we expect some **dynamical evolution** for the fast variables

$$x_i(t+1) = g_i\left(\{x_j\}, t\right)$$

Dynamical evolution of prices $p_i(t)$ and productions $\lambda_i(t)$? Following **rule of thumb**

- ⋆ If a firm is profitable, it will increase its output

$$\lambda_i(t+1) = \lambda_i(t)\left(1 + \beta \frac{P_i(t)}{z_i p_i(t) \lambda_i(t)}\right)$$

  where $P_i(t)$ is the profit of firm i at time t.

- ⋆ If a firm has excess supply, it will decrease its prices

$$p_i(t+1) = p_i(t)\left(1 - \alpha \frac{e_i(t)}{z_i \lambda_i(t)}\right)$$

  where $e_i(t)$ is the production surplus of firm i at time t.

Getting the causality right (1)

We want also to enforce **causality on trades and productions** i.e

- ⋆ Goods produced at time t cannot be bought or consumed before time $t+1$
- ⋆ Goods bought at time t cannot be used for production before time $t+1$

This leads to the **drop of the market clearing** hypothesis and the proper **rewriting of profit and surplus**

$$e_i(t) = \overbrace{z_i \lambda_i(t-1)}^{\text{production}} - \overbrace{\sum_j Q_{ji}(t)}^{\text{consumption}}$$

$$P_i(t) = \underbrace{\sum_j p_i(t) Q_{ji}(t)}_{\text{money earned}} - \underbrace{\sum_j p_j(t-1) Q_{ij}(t-1)}_{\text{money spent}}$$

Getting the causality right (1)

We want also to enforce **causality on trades and productions** i.e

- ⋆ Goods produced at time t cannot be bought or consumed before time $t+1$
- ⋆ Goods bought at time t cannot be used for production before time $t+1$

This leads to the **drop of the market clearing** hypothesis and the proper **rewriting of profit and surplus**

$$e_i(t) = \overbrace{z_i \lambda_i(t-1)}^{\text{production}} - \overbrace{\sum_j Q_{ji}(t)}^{\text{consumption}}$$

$$P_i(t) = \underbrace{\sum_j p_i(t) Q_{ji}(t)}_{\text{money earned}} - \underbrace{\sum_j p_j(t-1) Q_{ij}(t-1)}_{\text{money spent}}$$

Getting the causality right (2)

To enforce our dynamical rules, we need to **evaluate** $Q_{ji}(t)$. We do not have any information on the production target of other firms so we need an estimation

$$Q_{ji}(t) = J_{ji}\lambda_j(t)\left(\frac{\lambda_j(t)}{\lambda_j(t-1)}\right)^q \quad , \quad q \in [-1, 1]$$

- ⋆ $q = -1$, reverting behaviour,
- ⋆ $q = 0$, myopic behaviour,
- ⋆ $q = 1$, trend following.

To enforce our dynamical rules, we need to **evaluate** $Q_{ji}(t)$. We do not have any information on the production target of other firms so we need an estimation

$$Q_{ji}(t) = J_{ji}\lambda_j(t)\left(\frac{\lambda_j(t)}{\lambda_j(t-1)}\right)^q \quad , \quad q \in [-1, 1]$$

- ⋆ $q = -1$, reverting behaviour,
- ⋆ $q = 0$, myopic behaviour,
- ⋆ $q = 1$, trend following.

Since we enforced causality on trades and productions, the evolution for prices and productions id given by

$$p_i(t+1) = h_i^p\left(\{p_j, \lambda_j\}, t, t-1; \{\alpha, q, z_j, J_{ij}, \theta_j\}\right) \tag{1}$$

$$\lambda_i(t+1) = h_i^\lambda\left(\{p_j, \lambda_j\}, t, t-1; \{\beta, q, z_j, J_{ij}, \theta_j\}\right) \tag{2}$$

over a fixed configuration of z_j, J_{ij}, θ_j, α, β...

Equilibrium and resilience

Existence of equilibrium (1)

Does an equilibrium for prices and productions exist? If it does, coincides with the **fixed points of h_i^λ and h_i^p**.

More intuitively, **equilibrium** is reached when both **market clearing** (zero surplus) and **null profit** are fulfilled

$$e_i(t) = 0 \iff M^t \vec{\lambda}_{eq} = \vec{\xi}_{eq}$$
$$P_i(t) = 0 \iff M\vec{p}_{eq} = \vec{V}$$

with $\vec{V}$ vector of workforce, ξ vector of consumptions and $M_{ij} = z_i\delta_{ij} - J_{ij}$. **Positive solutions** iif $\epsilon = \phi(\{z_i\}, \{J_{ij}\}) > 0$

Existence of equilibrium (1)

Does an equilibrium for prices and productions exist? If it does, coincides with the **fixed points of h_i^λ and h_i^p**.
More intuitively, **equilibrium** is reached when both **market clearing** (zero surplus) and **null profit** are fulfilled

$$e_i(t) = 0 \iff M^t \vec{\lambda}_{eq} = \vec{\xi}_{eq}$$
$$P_i(t) = 0 \iff M\vec{p}_{eq} = \vec{V}$$

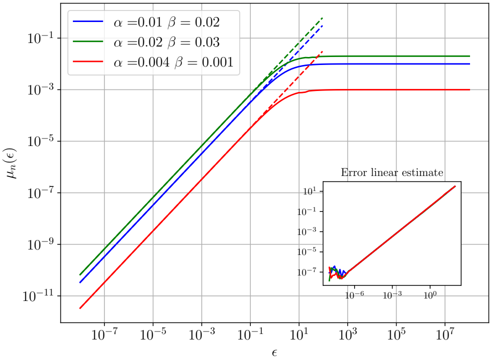
with $\vec{V}$ vector of workforce, ξ vector of consumptions and $M_{ij} = z_i\delta_{ij} - J_{ij}$. **Positive solutions** iif $\epsilon = \phi(\{z_i\}, \{J_{ij}\}) > 0$

Existence of equilibrium (1)

Does an equilibrium for prices and productions exist? If it does, coincides with the **fixed points of** h_i^λ **and** h_i^p.
More intuitively, **equilibrium** is reached when both **market clearing** (zero surplus) and **null profit** are fulfilled

$$e_i(t) = 0 \iff M^t \vec{\lambda}_{eq} = \vec{\xi}_{eq}$$
$$P_i(t) = 0 \iff M \vec{p}_{eq} = \vec{V}$$

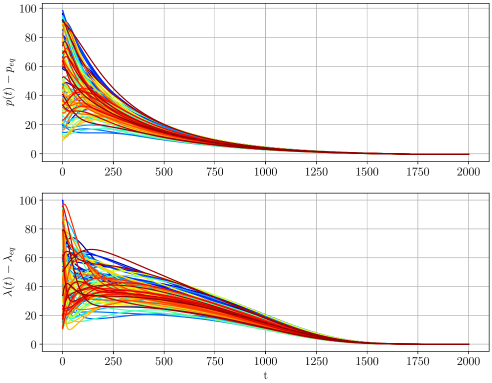
with $\vec{V}$ vector of workforce, ξ vector of consumptions and $M_{ij} = z_i \delta_{ij} - J_{ij}$. **Positive solutions** iif $\epsilon = \phi(\{z_i\}, \{J_{ij}\}) > 0$



Quantifying the relaxation time through mathematical analysis

$$\tau_{relax}(\epsilon) \sim \left| \frac{1}{\mu_n(\epsilon)} \right|$$

Relaxation towards equilibrium for $\epsilon > 0$

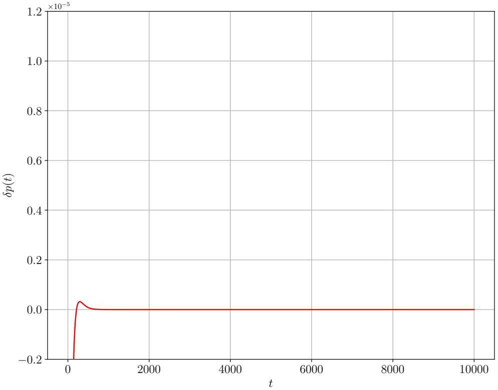


Marginal stability

Marginal stability: **equilibrium exists** but it takes an **infinite** amount of **time** to reach it.

$$\tau_{relax}(\epsilon) \sim \left|\frac{1}{\mu_n(\epsilon)}\right| \underset{\epsilon\to 0}{\sim} \frac{\rho_n}{\epsilon \min(\alpha, \beta)}$$

and $\boxed{\tau_{relax}(\epsilon) \to \infty \text{ for } \epsilon \to 0}$



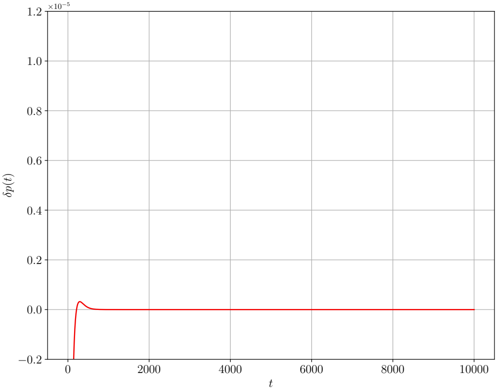
Light Red curve: ϵ_1, Dark Red curve: ϵ_2, Dark curve: ϵ_3 where $\epsilon_1 > \epsilon_2 > \epsilon_3$

Marginal stability

Marginal stability: **equilibrium exists** but it takes an **infinite** amount of **time** to reach it.

$$\tau_{relax}(\epsilon) \sim \left|\frac{1}{\mu_n(\epsilon)}\right| \underset{\epsilon\to 0}{\sim} \frac{\rho_n}{\epsilon \min(\alpha, \beta)}$$

and $\boxed{\tau_{relax}(\epsilon) \to \infty \text{ for } \epsilon \to 0}$



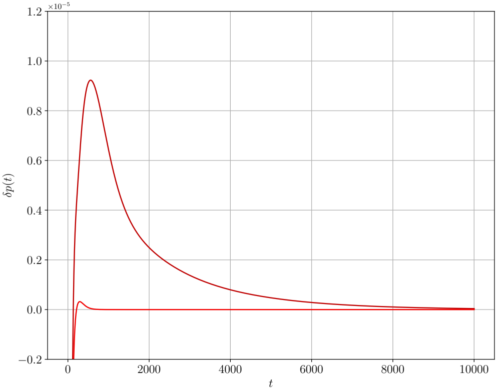
Light Red curve: ϵ_1, Dark Red curve: ϵ_2, Dark curve: ϵ_3 where $\epsilon_1 > \epsilon_2 > \epsilon_3$

Marginal stability

Marginal stability: **equilibrium exists** but it takes an **infinite** amount of **time** to reach it.

$$\tau_{relax}(\epsilon) \sim \left|\frac{1}{\mu_n(\epsilon)}\right| \underset{\epsilon\to 0}{\sim} \frac{\rho_n}{\epsilon \min(\alpha,\beta)}$$

and $\boxed{\tau_{relax}(\epsilon) \to \infty \text{ for } \epsilon \to 0}$



Light Red curve: ϵ_1, Dark Red curve: ϵ_2,

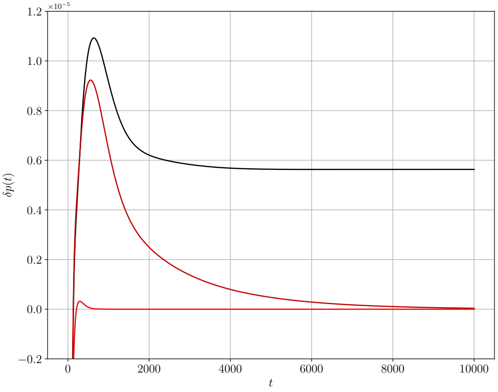
Dark curve: ϵ_3 where $\epsilon_1 > \epsilon_2 > \epsilon_3$

Marginal stability

Marginal stability: **equilibrium exists** but it takes an **infinite** amount of **time** to reach it.

$$\tau_{relax}(\epsilon) \sim \left| \frac{1}{\mu_n(\epsilon)} \right| \underset{\epsilon \to 0}{\sim} \frac{\rho_n}{\epsilon \min(\alpha, \beta)}$$

and $\boxed{\tau_{relax}(\epsilon) \to \infty \text{ for } \epsilon \to 0}$



Light Red curve: ϵ_1, Dark Red curve: ϵ_2, Dark curve: ϵ_3 where $\epsilon_1 > \epsilon_2 > \epsilon_3$

- ⋆ For small ϵ, nonresilient economies...
- ⋆ For small ϵ, aftermaths of shocks linger and accumulate for a potentially infinite amount of time.
- ⋆ If relaxation time is infinite, the configuration of "slow" variables may not be frozen anylonger: issue of rewiring, technological advances etc...

Resilience and shock propagation

- ⋆ For small ϵ, nonresilient economies...
- ⋆ For small ϵ, aftermaths of shocks linger and accumulate for a potentially infinite amount of time.
- ⋆ If relaxation time is infinite, the configuration of "slow" variables may not be frozen anylonger: issue of rewiring, technological advances etc...

Resilience and shock propagation

- ⋆ For small ϵ, nonresilient economies...
- ⋆ For small ϵ, aftermaths of shocks linger and accumulate for a potentially infinite amount of time.
- ⋆ If relaxation time is infinite, the configuration of "slow" variables may not be frozen anylonger: issue of rewiring, technological advances etc...

Conclusion

Conclusion and perspectives

- ⋆ Here multi-firm ABM $\longleftrightarrow$ multi-household DSGE-ABM crossover
- ⋆ Free evolution of supply chain network across time: how does it affect "fast" variables?
- ⋆ Policy making? Effects on the network (embargo, carbon footprint penalties...), effects on state variables (incentives to buy more from certain firms...)

Conclusion and perspectives

- ⋆ Here multi-firm ABM $\longleftrightarrow$ multi-household DSGE-ABM crossover
- ⋆ Free evolution of supply chain network across time: how does it affect "fast" variables?
- ⋆ Policy making? Effects on the network (embargo, carbon footprint penalties...), effects on state variables (incentives to buy more from certain firms...)

Conclusion and perspectives

- ⋆ Here multi-firm ABM $\longleftrightarrow$ multi-household DSGE-ABM crossover
- ⋆ Free evolution of supply chain network across time: how does it affect "fast" variables?
- ⋆ Policy making? Effects on the network (embargo, carbon footprint penalties...), effects on state variables (incentives to buy more from certain firms...)

BACK-UP

Discrete dynamical system

$$\lambda_i(t+1) = \lambda_i(t) - \beta \frac{\lambda_i(t)}{z_i p_i(t)} \left(\sum_{j=1}^{n} J_{ij} p_j(t-1) + V_i \right) + \frac{\beta}{z_i} \sum_{j=1}^{n} J_{ji} \lambda_j(t) \left(\frac{\lambda_j(t)}{\lambda_j(t-1)} \right)^q + \beta \frac{\mu \theta_i}{z_i p_i(t)}$$

$$p_i(t+1) = \alpha \frac{p_i(t)}{z_i \lambda_i(t)} \left(\sum_{j=1}^{n} J_{ji} \lambda_j(t) \left(\frac{\lambda_j(t)}{\lambda_j(t-1)} \right)^q + \frac{\mu \theta_i}{p_i(t)} \right) + p_i(t) \left(1 - \alpha \frac{\lambda_i(t-1)}{\lambda_i(t)} \right)$$

with $V_i = p_0 J_{i0}$ and $\mu = \frac{1}{\sum_i \theta_i}$.

Relaxation towards equilibrium

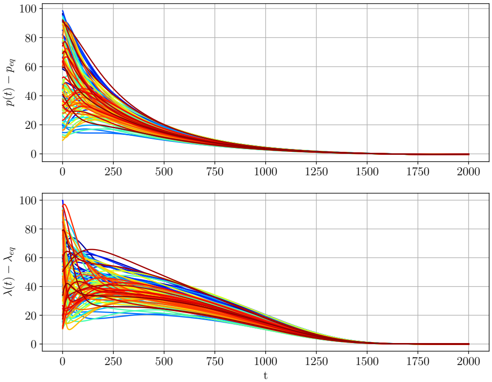


Figure: Simulation for $n = 100$ firms on a 3-regular network (3 neighbours), $\alpha = 0.01$ and $\beta = 0.05$, $q = 1$.

Phase separation in parameter space