Introduction to Financial Networks and Systemic Risk

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New Analytical Tools and Techniques for Economic Policymaking
New Approaches to Economic Challenge
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Point of View

- Financial Systemic Risk (FSR): intertwines ideas from economics, social policy, finance, physics, computer science, mathematics, probability and statistics, other sciences.
- Personal perspective: Provide mathematical modeling framework based on network science for transmission of damaging shocks through financial systems.

Contagion! Systemic Risk in Financial Networks

Definition

systemic risk (SR): risk that default or stress of one or more financial institutions ("banks") will trigger default or stress of further banks, leading to large scale cascades of system failures and large negative impact on external economy.

Main Aim

1. *To crystallize a basic modelling structure for systemic risk.*
2. *To ensure mathematical tractability, scalability and reality for actual finance network specifications.*

Goals of this Minicourse

1. Introduction of IRFN framework, providing flexible and scalable architecture for modelling many complex network characteristics, in particular banks with arbitrary types.
2. Introduce cascade analysis for an economically important family of models extending the EN 2001 framework to include *partial fractional recovery* of defaulted interbank assets.
3. A rigorous and general characterization of the first cascade step in IRFN default models, in the limit $N \to \infty$, including general decorrelation results stemming from the *locally tree-like independence (LTI)* property.
4. Introduction of *simplified analytical* results on the large N asymptotics for default cascades. Tractable recursive formulas for *cascade equilibria* are derived through a new (non-rigorous) style of argument.

Benefits of IRFN

The IRFN construction provides several benefits compared to the "configuration graph" RFN construction.

1. Node type has a direct and basic financial interpretation: logically, a node's connectivity degree is dependent on its type. Type is a more intuitive and general notion than node degree, and better suited to SR modelling. Node types can encode an unlimited range of node characteristics.
2. Node type makes better financial sense than node degree as conditioning random variables to determine system dependencies. That is, assuming random balance sheets and exposures are independent conditioned on node types is better justified than assuming their independence conditioned on node degrees.
3. Inhomogeneous RFNs are easier to explain and analyze than "configuration" RFNs.

How Complex is the System?

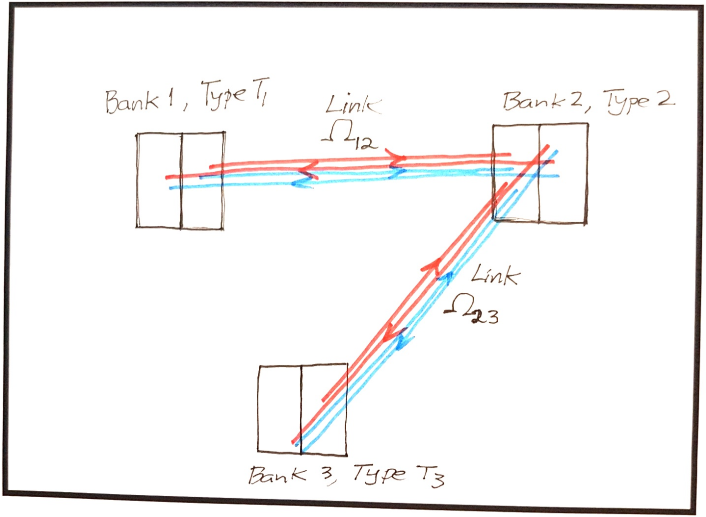
1. "nodes": "banks" are a diverse collection of large and small, regulated and unregulated hierarchical institutions.
2. " links": an interbank exposure is a complex portfolio of complex bilateral financial contracts, changing intra-daily.
3. Bank strategies: highly dynamic; the result of decision making distributed across all the subdivisions of the bank; made under great uncertainty.

How Complex is the System (ctd)?

4. Time scales: range from microseconds to decades.
5. Global banking systems are strongly connected.
6. "banks are special" (Corrigan, 1982): they play a critical systemic role at the heart of the much larger macroeconomy.
7. System is opaque: SR data is typically aggregated and disaggregated links are usually invisible.
8. System is controlled/monitored by central banks.

What is the System?

System as a network of nodes v ("banks") with balance sheets B interconnected by contractual obligations Ω_{vw} ("links").



Node Types: Banks

Assets	Liabilities
inter-agent assets $\bar{Z}$	inter-agent debt $\bar{X}$
external illiquid assets $\bar{A}$	external debt $\bar{D}$
external liquid assets $\bar{C}$	equity $\bar{E}$

Table: Stylized bank balance sheet.

Node Types: Funds

Assets	Liabilities
inter-agent assets $\bar{Z}$	equity $\bar{E}$
external illiquid assets $\bar{A}$	
external liquid assets C	

Table: Stylized investment fund balance sheet.

Node Types: Firms

Assets	Liabilities
external physical assets $\bar{\mathrm{A}}$	inter-agent debt $\bar{\mathrm{X}}$
	external debt D
	equity E

Table: Stylized firm (investment asset) balance sheet.

Network Representation of a Diversified Economy

The financial system at a moment in time is a multidimensional random variable (RV) with two levels of structure.:

1. Level I is called the *skeleton graph*. The directed random graph (DRG) where:
   - Nodes are financial institutions, each with node type label T;
   - Directed edges represent the existence of a significant exposure of one node to another.
2. Level II specifies the *balance sheets* B of the agents, including the inter-node exposures Ω. The RVs of Level II have some conditional independence, conditioned on knowledge of the skeleton graph.

Popular RG constructions in the literature: (1) Scale-free models; (2) configuration models; (3) inhomogeneous RG models.

Node Types

Typical classification scheme for the nodes in system.

1. Label nodes of the system by multi-indices $T = (c, x, t, s) \in \mathcal{T}$ where $\mathcal{T}$ is type space with finite size $|\mathcal{T}| = \mathbb{T}$.
2. Country index c, taken from the index set $[N_c] := \{1, 2, \dots, N_c\}$, N_c is the number of countries.
3. Geographical index x taking values in $[N_x(c)]$ depending on the country.
4. Category of node $t \in [3] := \{1, 2, 3\}$ where 1 denotes a "bank", 2 denotes a "fund" and 3 denotes a "firm".
5. Size $s \in \{l, m, s\}$ for "large", "medium", "small".

Remarks on Model Implementation

1. Missing Data? Randomize!
2. Each country can be modeled quite differently, depending on one's perspective.
3. The "home country" model for $c = 1$ may be quite determined and precise.
4. "ROW - Rest of the World" model for $c > 1$ may be very stylistic and highly randomized.

The Skeleton: an Inhomogeneous Random Graph

Fix N = total number of nodes in the system; label nodes by $v \in [N]$.

1. Each node v has a random type $T_v = (c_v, x_v, t_v, s_v, \tau_v) \in \mathcal{T}$, drawn independently from the probability measure $\mathbb{P}(T)$
2. For each pair of nodes $v \neq w$, the Bernoulli RVs $I_{vw} \in \{0,1\}$ determine which potential links have "large exposures". These RVs are conditionally independent, with

$$\mathbb{P}(I_{vw} = 1 \mid T_v = T, T_w = T') = \frac{\kappa(T,T')}{N - 1 + \kappa(T,T')}$$

3. kernel $\kappa : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$ encodes counterparty preferences between banks.
4. Assumed N dependence leads to sparse networks for $N \to \infty$.

In/Out Degree Distribution

In/out degree of node v: $(\sum_w I_{wv}, \sum_w I_{vw})$.

Proposition

The $N \to \infty$ limit in distribution of the joint in/out degree distribution of each node v is a mixture of bivariate Poisson mixture distributions. The mixing variable is the node-type T which has the mixing weights $\mathbb{P}(T)$, and the bivariate Poisson distribution components each have mean parameters

$$(\sum_{T_w} \mathbb{P}(T_w)\kappa(T_w, T), \sum_{T_w} \mathbb{P}(T_w)\kappa(T, T_w)) \ ,$$

1. All components of $\bar{\mathrm{B}} = [\bar{\mathrm{Z}}, \bar{\mathrm{A}}, \bar{\mathrm{C}}, \bar{\mathrm{X}}, \bar{\mathrm{D}}, \bar{\mathrm{E}}]$ and $\bar{\Omega}$ are non-negative.
2. *accounting identities* are satisfied:

$$\bar{\mathrm{Z}}_v = \sum_w I_{wv}\bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w I_{vw}\bar{\Omega}_{vw}, \quad \sum_v \bar{\mathrm{Z}}_v = \sum_v \bar{\mathrm{X}}_v,$$
$$\bar{\mathrm{Z}}_v + \bar{\mathrm{A}}_v + \bar{\mathrm{C}}_v = \bar{\mathrm{X}}_v + \bar{\mathrm{D}}_v + \bar{\mathrm{E}}_v \ .$$

3. We introduce *cash buffer* $\Xi_v^{(0)} := \bar{\mathrm{C}}_v + \delta \mathrm{C}_v$ which may be negative, in which case the node v is said to be *illiquid*.
4. Similarly, the *solvency buffer* $\Delta_v^{(0)} := \bar{\mathrm{E}}_v + \delta \mathrm{E}_v$ may be negative, in which case the node is said to be *insolvent*.

Balance Sheet Probabilities

In randomized models like EN 2001 and GK 2010, it is sufficient to specify distributions of Δ and Ω. Conditioned on the collection $\{T_v\}$, these form a fully independent collection of RVs.

1. Buffer distribution of Δ_v conditioned on T_v:

$$\rho_\Delta(x \mid T_v) = \frac{d}{dx} F_\Delta(x \mid T_v); \;\; F_\Delta(x \mid T_v) = \mathbb{P}(\Delta_v \le x \mid T_v)$$

2. Distribution of Ω_{vw} conditioned on T_v, T_w:

$$\rho_\Omega(x \mid T, T') = \frac{d}{dx} F_\Omega(x \mid T, T'); \; F_\Omega(x \mid T_v, T_w) = \mathbb{P}(\Omega_{vw} \le x \mid T_v$$

Basic Modelling Strategy

1. Define Cascade Mechanism (CM): shock transmission in system.
2. Shock system at time 0 and determine resultant cascade equilibrium.

Crisis Trigger and Buffers

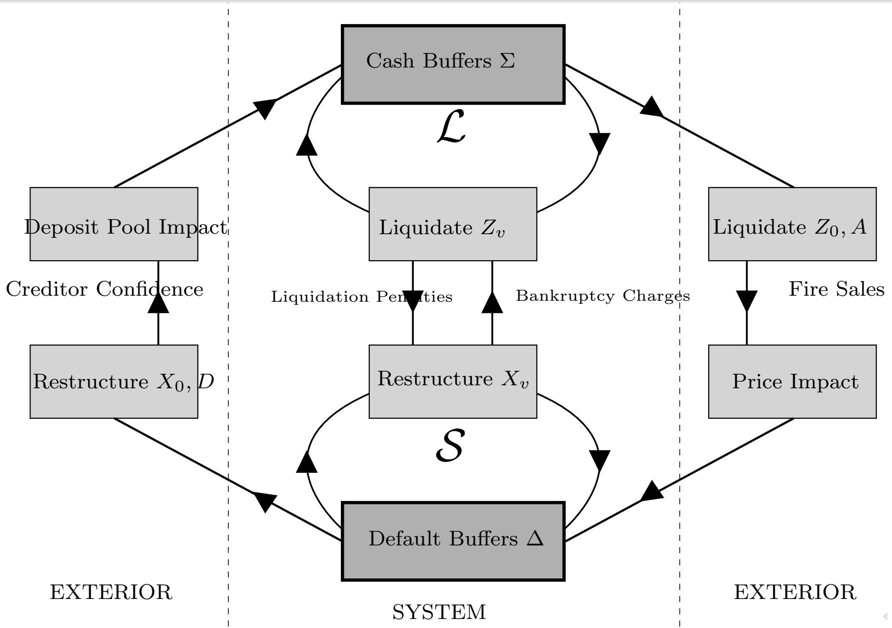
1. A *crisis trigger* at a moment in time, which we label by step $n = 0$, occurs when a shock $\delta \mathrm{B}$ to the balance sheets is sufficiently severe to put some nodes into a stressed state where not all of their balance sheet entries $\mathrm{B}^{(0)} = \bar{\mathrm{B}} + \delta \mathrm{B}$ are positive.
2. We introduce *cash buffer* $\Xi_v^{(0)} := \bar{\mathrm{C}}_v + \delta \mathrm{C}_v$ which may be negative, in which case the node v is said to be *illiquid*.
3. Similarly, the *solvency buffer* $\Delta_v^{(0)} := \bar{\mathrm{E}}_v + \delta \mathrm{E}_v$ may be negative, in which case the node is said to be *insolvent*.
4. The *cascade* that follows the crisis trigger for $n \geq 0$ is a step-wise dynamics for the collection of balance sheets $\mathrm{B}_v^{(n)}$ of the entire system as it tries to *resolve* these illiquid and insolvent nodes.

Channels of Systemic Risk (SR)

Five important channels of SR:

1. **Correlation** e.g. Subprime assets or deposit withdrawals
2. **Default Contagion** e.g. default of Lehman
3. **Liquidity Contagion** e.g. the freezing of repo markets
4. **Firesales or Market Illiquidity** e.g. sales of ABS
5. **Rollover or Creditor Confidence Risk** , e.g. wholesale depositor runs

Systemic Risk Schematic



Default Cascade Mechanism

1. Cascade mechanisms (CMs) are stylized behaviours followed by nodes when it becomes known that a crisis has been triggered.
2. Behaviour is highly non-linear, reflecting that during a crisis, "stressed" nodes will take strong emergency actions.
3. "Business as usual" in which nodes react smoothly/linearly to small changes is not applicable to many nodes during crisis.
4. Instead, we assume that during crisis, "not-yet stressed" nodes adopt a "do nothing/wait and see" crisis management strategy.
5. From a systemic perspective, cascades arise when nodes' behaviour have negative impact on other nodes.
6. In simple default cascades $\mathcal{S}$, only the impacted solvency buffer $\Delta_v^{(n)} = \Delta_v^{(0)} - \sum_w S_{wv}^{(n-1)}$ is relevant.

Eisenberg-Noe 2001 and Gai-Kapadia 2010 Default Cascade

1. External debt D is senior to interbank debt X.
2. No bankruptcy losses (or, GK: 100% losses).
3. Exposure Ω_{vw}: what bank v owes bank w.
4. Default Buffer: $\Delta = \mathrm{A} + \mathrm{Z} + \bar{\mathrm{C}} - \mathrm{D} - \mathrm{X}$.
5. Limited Liability: bank v is defaulted if and only if $\Delta_v \leq 0$.
6. Buffers like Δ provide a safety margin to absorb shocks.

Vector and matrix notation

For vectors $x = [x_v]_{v=1,\dots,N}, y = [y_v]_{v=1,\dots,N} \in \mathbb{R}^N$

$$\begin{aligned} x \le y \quad &\text{means} \quad \forall\, v,\ x_v \le y_v, \\ \min(x,y) = x \wedge y \quad &= \quad [\min(x_v, y_v)]_{v=1,\dots,N} \\ \max(x,y) = x \vee y \quad &= \quad [\max(x_v, y_v)]_{v=1,\dots,N} \\ (x)^+ \quad &= \quad \max(x, 0), \\ (x)^- \quad &= \quad \max(-x, 0) \end{aligned}$$

For $x \le y$, the hyperinterval $[x, y]$ is $\{z : x \le z \le y\}$. Any hyperinterval, with the above operations $\wedge, \vee$, is a complete lattice[1].

[1] A "lattice" (partially ordered set with "meet" $\vee$ and "join" $\wedge$) that is closed under sup and inf.

EN 2001 Equilibrium

- $\mathbf{p} = [p_1, \ldots, p_N]$: fractional amounts available to pay internal debts to creditor banks.
- Clearing condition:

$$\mathrm{X} \cdot *\mathbf{p}^* = \mathrm{X} \wedge \left(\mathrm{A} + \mathrm{C} + \Omega^T * \mathbf{p}^* - \mathrm{D}\right)^+$$

- Equilibrium buffers:

$$\Delta^* = \mathrm{A} + \mathrm{C} + \Omega^T * \mathbf{p}^* - \mathrm{D} - \mathrm{X}$$

EN Main Theorem

Theorem

Corresponding to every financial system $(\bar{\mathrm{A}}, \bar{\mathrm{C}}, \bar{\mathrm{Z}}, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega})$ *satisfying EN 2001/GK 2010 Assumptions there exists a greatest and a least clearing vector* p^+ *and* p^-.

Proof

Knaster-Tarski Fixed Point Theorem states: "the fixed point set of a monotone mapping on a complete lattice is a complete lattice".

Note:

1. F^{EN} is monotonic: $x \leq y$ implies $F^{EN}(x) \leq F^{EN}(y)$.
2. Since also $F^{EN}(0) \geq 0$ and $F^{EN}(\bar{\mathrm{X}}) \leq \bar{\mathrm{X}}$, it maps the hyperinterval $[\mathbf{0}, \bar{\mathrm{X}}]$ into itself.
3. $[\mathbf{0}, \bar{\mathrm{X}}]$ is a complete lattice.

Conclude that set of fixed points of the mapping F^{EN}, is a complete lattice, hence nonempty, and with maximum and minimum elements p^+ and p^-.

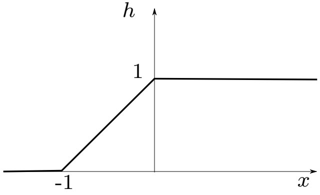
Default Buffer Mapping

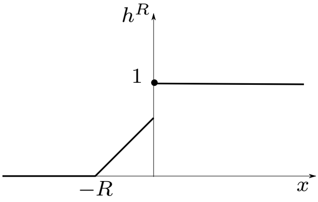
1. Default buffer after n cascade step $\Delta_w^{(n)}$ satisfies recursion

$$\Delta_w^{(n)} = \Delta_w^{(0)} - \sum_v \Omega_{vw} \left(1 - h^R(\Delta_v^{(n-1)}/\mathrm{X}_v)\right)$$

2. Threshold function gives fractional recovered value of defaulted interbank assets:

$$\begin{aligned} h(x) &= (x+1)^+ - (x)^+ \\ h^R(x) &= Rh(x/R) + (1-R)\mathbf{1}(x>0) \end{aligned}$$





Fractional Recovery Assumption

"Zero Recovery" and "Zero Bankruptcy Charges" are both inadequate assumptions for SR. We assume:

1. External debt D is senior to interbank debt X and all interbank debt is of equal seniority;
2. Bankruptcy charges are in proportion to the negative part of the impacted solvency buffer.

That is, at step n of the cascade

$$\text{bankruptcy costs} = (1/\lambda - 1)\max(-\Delta^{(n)}, 0) \ ,$$

for some parameter $\lambda \in (0, 1]$.

Insolvency Random Variables

1. Random variables

$$\mathcal{D}_v^{(n)} = \min\left(1, \frac{\max(-\Delta_v^{(n)}, 0)}{\lambda \bar{X}_v}\right) \in [0, 1]$$

   measure the *loss fraction on interbank debt* of each bank at step n.

2. The insolvency level of any bank w at step n now influences the default shock transmitted to another bank v:

$$S_{wv}^{(n)} = I_{wv} \bar{\Omega}_{wv} \mathcal{D}_w^{(n)}$$

Cascade Steps

The cascade mapping at step $n = 0$ is

$$
\begin{aligned}
\mathcal{D}_v^{(0)} &= \min\left(1, \frac{\max(-\Delta_v^{(0)}, 0)}{\lambda \bar{X}_v}\right) \\
S_{wv}^{(0)} &= I_{wv}\bar{\Omega}_{vw}\mathcal{D}_w^{(0)}; \quad S_v^{(0)} := \sum_{w \neq v} S_{wv}^{(0)} \\
\Delta_v^{(1)} &= \Delta_v^{(0)} - S_v^{(0)}.
\end{aligned}
$$

The cascade mapping at step $n \geq 0$ is

$$
\begin{aligned}
\mathcal{D}_v^{(n)} &= \min\left(1, \frac{\max(-\Delta_v^{(n)}, 0)}{\lambda \bar{X}_v}\right), \\
S_{wv}^{(n)} &= I_{wv}\bar{\Omega}_{vw}\mathcal{D}_w^{(n)}; \quad S_v^{(n)} := \sum_{w \neq v} S_{wv}^{(n)} \\
\Delta_v^{(n+1)} &= \Delta_v^{(0)} - S_v^{(n)} .
\end{aligned}
$$

Asymptotic Analysis

Lemma

1. *The characteristic function of the interbank debt* $\bar{\mathrm{X}}_1 = \sum_{w \neq v} \bar{\Omega}_{wv}$ *of bank* 1*, conditioned on* $T_1 = T \in \mathcal{T}$*, has the* $N \to \infty$ *limiting behaviour:*

$$\hat{f}^{(N)}_{\mathrm{X}}(k \mid T) := \mathbb{E}^N\Big(\prod_{w \neq 1} e^{ikI_{w1}\bar{\Omega}_{w1}} \mid T\Big) = \hat{f}_X(k \mid T)(1 + O(N^{-1}))$$

$$\hat{f}_X(k \mid T) = \exp\left[\sum_{T'} \mathbb{P}(T')\kappa(T', T)(\hat{f}_{\bar{\Omega}}(k \mid T', T) - 1)\right]$$

where convergence of $\log \hat{f}$ *is in* $L_2[0, \infty)$

2. *Any finite collection of interbank debt RVs* $\bar{\mathrm{X}}_v$ *is independent in the* $N \to \infty$ *limit.*

Shock Distribution

Conjecture

The conditional characteristic function of the total default shock $S^{(0)}_1 := \sum_{w\neq 1} I_{w1}\mathrm{X}_{w1}\mathcal{D}^{(0)}_w$ *transmitted to bank* 1 *in step* 0*:*

$$\hat{f}^{(N)}_{S^{(0)}}(k \mid T) = \exp\left[\int_0^\infty (e^{iku}-1)\mu^{(0)}(u \mid T)\mathrm{d}u\right](1+O(N^{-1})) \quad (1)$$

$$\mu^{(0)}(u \mid T) = \sum_{T'} \mathbb{P}(T')\kappa(T',T)\lambda\int_0^\infty R(u,y \mid T,T')\rho_{\Delta^{(0)}}(-\lambda y \mid T')\mathrm{d}y$$

$$R(u,y \mid T,T') = \frac{\mathbf{1}(y>u)y}{(y-u)^2}\int_{y-u}^\infty x\rho_{\mathrm{X}}(x \mid T')\rho_\Omega\left(\frac{ux}{y-u} \mid T',T\right)\mathrm{d}x$$
$$+\text{another term}$$

Heuristic Argument (note the unproven step)

$$\log \mathbb{E}^N(\prod_{w\neq 1} e^{ikI_{w1}\mathrm{X}_{w1}\mathcal{D}_w^{(0)}} \mid T) \quad \overset{?}{\sim} \quad (N-1)\log \mathbb{E}^{(N)}(e^{ikI_{21}\mathrm{X}_{21}\mathcal{D}_2^{(0)}} \mid T) + O$$

$$= (N-1)\log \mathbb{E}^{(N)}(e^{ikG(X,Y,Z)}) + O(N^{-1})$$

Here $X = \sum_{w\neq 1,2} I_{w1}\bar{\mathrm{X}}_{w1}\mathcal{D}_w^{(0)}, Y = (-\Delta_2^{(0)}/\lambda)^+, Z = I_{21}\bar{\Omega}_{21}$, and probability is conditioned on $T_1 = T$.

$$G(X,Y,Z) = \min(Z, ZY/(Z+X))$$

Now write $N-1 = \epsilon^{-1}$ and

$$g(k,\epsilon) = \sum_{T'} \mathbb{P}(T')\frac{\kappa(T',T)}{1+\epsilon\kappa(T',T)}\mathbb{E}(e^{ikG^\epsilon(X,Y,Z)})$$

We have a general L_2-bound $\epsilon^{-1}\log(1+\epsilon g(k,\epsilon)) - g(k,0) = O(\epsilon)$ for ϵ small.

Condensed Cascade Mapping

Precompute the function

$$\tilde{R}(k,\tilde{k}\mid T,T') := \mathbb{P}(T')\iint_{\mathbb{R}_+^2}(e^{iku}-1)e^{-i\lambda\tilde{k}y}R(u,y\mid T,T')\mathrm{d}y\mathrm{d}u$$

Suppose we have the CF $\hat{f}_{\Delta^{(n)}}(k\mid T)$. To compute $\hat{f}_{\Delta^{(n+1)}}(k\mid T)$:

1. CF of total default shock $S_1^{(n)}$:

$$\hat{f}_{S^{(n)}}(k\mid T) = \exp\left[\int_{-\infty}^{\infty}\sum_{T'}\tilde{R}(k,\tilde{k}\mid T,T')\ \hat{f}_{\Delta^{(n)}}(k\mid T')\ \mathrm{d}k\right]$$

2. CF of impacted default buffer $\Delta_1^{(n+1)} = \Delta_1^{(0)} - S_1^{(n)}$:

$$\hat{f}_{\Delta^{(n+1)}}(k\mid T) = \hat{f}_{\Delta^{(0)}}(k\mid T)\hat{f}_{S^{(n)}}(-k\mid T)$$

Computational Complexity

1. Choose suitable truncation parameters $\Lambda = \delta M, M$ such that precomputing the function $\tilde{R}(k, \tilde{k} \mid T, T')$ over the grid $\Gamma = \delta\{-M + 1/2, -M + 3/2, \dots, M - 3/2, M - 1/2\}^2 \subset \mathbb{R}^2$ provides sufficient accuracy. Using Simpson's rule, this is potentially as much as $K \times O(M^5 \times \mathbb{T}^2)$ flops with $|\mathcal{T}| = \mathbb{T}$.
2. Each cascade step maps the $2M \times N$ dimensional vector $\hat{f}_{\Delta^{(n)}}$ to the exponential of a $[2M \times \mathbb{T}, 2M \times \mathbb{T}]$ matrix multiplication $\hat{f}_{S^{(n)}} = \exp[\tilde{R} * \hat{f}_{\Delta^{(n)}}]$, followed by a Hadamard (element-wise) product $\hat{f}_{\Delta^{(n+1)}} = \hat{f}_{S^{(n)}} \cdot * \hat{f}_{\Delta^{(0)}}$.
3. A single cascade step is thus of order $O(M^2 \times \mathbb{T}^2)$ flops, followed by $O(M \times \mathbb{T})$ exponentiations.

Global Financial System

A stylized implementation scheme for the *global financial system* might involve a random graph with nodes something like this:

1. The *country index* $c \in [N_c] := \{1, 2, \ldots, N_c\}$, where N_c is the number of countries.
2. The *size* is $s \in [3] := \{l, m, s\}$, for "large", "medium", and "small".
3. Geographical labels and category labels can also be included.
4. Let total number of banks in the real world system be $\bar{N} = \sum_T \bar{N}_T$. Since there are a lot of small banks in the world, assume $\bar{N} \sim 10000$. The type probabilities are the empirical probabilities $\mathbb{P}(T) = \bar{N}_T / \bar{N}$.
5. For any $k \geq 1$, consider the network with $N = k\bar{N}$ by taking k copies of each bank. Then the large N limit of the real world system can be interpreted as the $k \to \infty$ limit.

Probability Mapping Kernel $\kappa(T, T')$

Typical two-country sector, Canada and France. $N_{CA} \sim 30$ banks.

T \ T'	'CA-l'	'CA-m	'CA-s'	$\cdots$	'FR-l'	'FR-m'	'FR-s'	$\cdots$
'CA-l'	9000	400	100	$\cdots$	40	5	2	$\cdots$
'CA-m'	400	200	50	$\cdots$	5	4	0	$\cdots$
'CA-s'	80	50	10	$\cdots$	2	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
'FR-l'	35	4	2	$\cdots$	7000	300	120	$\cdots$
'FR-m'	6	3	0	$\cdots$	350	150	40	$\cdots$
'FR-s'	1.5	0	0	$\cdots$	130	45	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table: The probability mapping kernel $\kappa(T, T')$: Canada and France. $\kappa(T, T')\mathbb{P}(T')$ represents the expected number of exposures of a bank of the given row type to banks of the given column type.

Multiple Contagion Channels

Suppose banks have both a solvency buffer Δ_v and a liquidity buffer Σ_v. Bank v has possibility to transmit both default shocks S_{vw} and liquidity shocks $\tilde{S}_{vw}$ to its counterparties w.

$$\begin{aligned}
\mathcal{D}_v^{(n)} &= \min\left(1, \frac{\max(-\Delta_v^{(n)},0)}{\lambda\bar{X}_v}\right), \widetilde{\mathcal{D}}_v^{(n)} = \min\left(1, \frac{\max(-\Sigma_v^{(n)},0)}{\tilde{\lambda}\bar{X}_v}\right) \\
S_{wv}^{(n)} &= I_{wv}\bar{\Omega}_{wv}\mathcal{D}_w^{(n)}; \quad S_v^{(n)} := \sum_{w\neq v} S_{wv}^{(n)} \\
\tilde{S}_{wv}^{(n)} &= I_{vw}\bar{\Omega}_{vw}\widetilde{\mathcal{D}}_w^{(n)}; \quad \tilde{S}_v^{(n)} := \sum_{w\neq v} \tilde{S}_{wv}^{(n)} \\
\Delta_v^{(n+1)} &= \Delta_v^{(0)} - S_v^{(n)}, \quad \Sigma_v^{(n+1)} = \Sigma_v^{(0)} - \tilde{S}_v^{(n)}
\end{aligned}$$

The probabilistic cascade mapping is then

$$\hat{f}_{\Delta,\Sigma}^{(n+1)} = \hat{f}_{\Delta,\Sigma}^{(0)} \cdot * \exp[R * \hat{f}_{\Delta,\Sigma}^{(n)}]$$

where now R denotes a function $R(k,\tilde{k};k',\tilde{k}' \mid T,T')$.

Simulations: Mean Cascade Size and Cascade Frequency

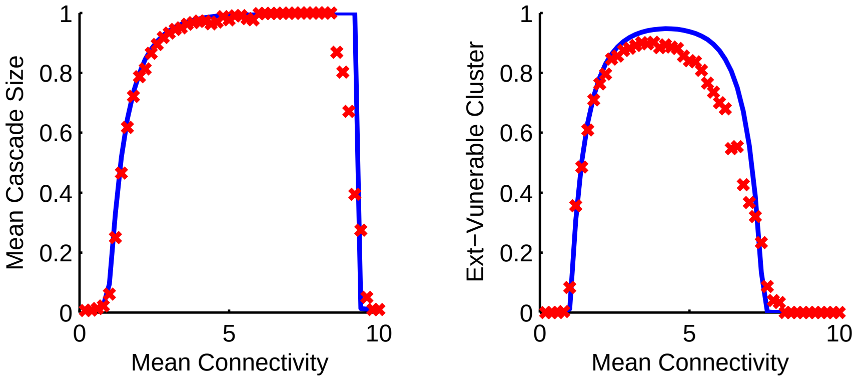


Figure: Mean fractional cascade size, and global cascade frequency as a function of mean degree z by large N analytics (blue curve) and by Monte Carlo simulation (red crosses).

Conclusions and Questions

1. Under some conditions to be understood better, “large N” cascade boils down to

$$\hat{f}_\Delta^{(n+1)} = \hat{f}_\Delta^{(0)} \cdot\!* \exp[\tilde{R} * \hat{f}_\Delta^{(n)}]$$

2. $\tilde{R}, \hat{f}_\Delta^{(0)}$ encode all relevant information of initial system.
3. Can an extended percolation theory prove the conjectured mappings?
4. We need now to bring model to data!
5. Much work remains: How do analytics compare to simulations on finite size networks?