Chapter 8

BILATERAL AND MULTILATERAL COMPARISONS OF PRODUCTIVITY IN INPUT-OUTPUT ANALYSIS USING ALTERNATIVE INDEX NUMBERS

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Introduction

The economic literature on productivity measurement has developed a number of alternative procedures for the estimation of productivity change. Diewert (1992*a*), for example, distinguishes six different approaches in the context of a very simple environment with one output and one input. Five of these approaches turn out to be completely equivalent, and the sixth approach is also equivalent under a particular assumption that is not required by the other methods. Although these results are very encouraging, productivity measurement becomes problematic as soon as more realistic cases are examined. In many-output and many-input cases, for example, an aggregation problem arises and in general there is no unique way to measure the rates of change of the entire sets of output and input quantities. Different solutions to this problem can lead to very different numerical results.

Each of the main theoretical approaches to productivity measurement can be implemented by alternative methods. These methods can be classified according to three major categories: econometric estimation of cost and production functions; non-parametric mathematical programming methods for the measurement of productivity and inefficiency; application of index numbers. This chapter focuses on the third of these methodologies and is based on the so-called “economic theory of index numbers”, which studies, in particular, the interrelationship between the aggregation procedure and the specification of functional forms under the hypothesis of efficient allocation of resources.

The aggregation problem is related to a number of questions: *i*) data, determining the level and the accuracy of the aggregation of elementary index numbers; *ii*) production conditions, affecting the functional forms and the sensitivity of index numbers with respect to technological change, returns to scale and external economies and diseconomies; *iii*) market conditions, affecting the economic information incorporated in the available data and, in particular, the price-cost ratios; *iv*) economic behaviour of firms, affecting the optimisation process and the interpretation of the observed prices and quantities; *v*) model specification, inherent to the choice of the most appropriate functional form of the index number. These issues are all extremely important for the identification of the correct methodology to be used for the aggregation procedure.

Notwithstanding the number of factors affecting productivity measurement, little work has been done so far to evaluate how much alternative index numbers of productivity can differ from each other. It is, nonetheless, of great importance to establish at least the range of possible numerical values of these index numbers and the degree of their representativeness. From the viewpoint of the analyses of productivity and their policy implications, it is important to assess the sensitivity of the results that are obtained through adopting different functional forms.

The chapter presents empirical evidence on the difference in productivity estimations that may be obtained by using alternative index numbers. Fixed- and flexible-weight index numbers have been employed to assess the sensitivity of the results; the extension of the methodology of bilateral index numbers to multilateral indexes for intertemporal and interspatial comparisons is also re-examined. Another novel feature of this chapter is the incorporation of index numbers into the input-output accounting system in order to estimate the indirect productivity levels that are incorporated into the intermediate input costs. The indirect productivity levels are compared with the more traditional measures of direct productivity components of cost differences. Our finding is that alternative productivity indexes can differ widely. Moreover, the conceptual definitions of direct and indirect productivity effects produce some surprising results.

The following section examines the choice of the functional form of index numbers and how the empirical results may be affected by this choice. In the third section, bilateral comparisons based on index numbers are used as building blocks for multilateral comparisons in an intertemporal and interspatial context. In the fourth section, the methodology discussed in the preceding sections is incorporated into the framework of input-output models in order to evaluate the direct and indirect effects of productivity changes. The following section presents the empirical results of comparison of alternative index numbers and definitions of direct and indirect productivity; the final section concludes.

Choice of functional forms of index numbers

Index number theory has developed three main approaches to establish criteria for the choice of index numbers: the "stochastic approach" – based on attributing a stochastic component to the elementary indices, estimating the mean of these indices and using their estimated standard errors to select the "best index" among competing alternatives; the "axiomatic" or "test approach" – based on the formulation of desirable properties that should be exhibited by the ideal index number; and the "economic approach" – which establishes an exact relationship between the functional forms of index numbers and those of the underlying technological relationships between output and input quantities or prices under the hypothesis of an efficient allocation of resources. From the point of view of economic interpretation of the changes observed, the third approach appears more promising than the others, although the assumptions about the optimising behaviour of firms are sometimes too strong.

One of the basic concepts on which the economic approach relies is that the price-taker producer minimises costs of production by choosing an optimal combination of input quantities for given levels of outputs and technical knowledge. In the multiple-output, multiple-input technology, producers are induced to adjust the combination of input quantities if they face changes in relative prices. In this situation, the measurement of aggregates of output and input quantities and input prices is, in general, problematic: the weights (given by prices for aggregating quantities and by quantities for aggregating prices) are different in the two observation points under comparison and, therefore, the empirical results may depend upon the chosen weights and the chosen functional form of the aggregating index number. The main problem in this approach is, therefore, to find the weights and, more generally, the functional form of the aggregating index number that is most consistent with the underlying technology and economic behaviour of the firm.

Since, in general, the "true" technological and behavioural conditions are not known, the corresponding "true" index numbers for the aggregate output and input quantities and prices, if these exist, are also unknown. In the general case, the best we can do is to take account of the conceptual relationship between the available functional forms of index numbers and the corresponding underlying technological and behavioural conditions and evaluate the differences in the respective empirical results. Moreover, under very restrictive hypotheses, we can, at most, indicate which index numbers can be used as the extremes of the interval of possible values of the unknown "true" index number. In this approach, index numbers can be viewed in a similary way to the theory of revealed preferences in the field of the consumer economics.

More specifically, in period or country τ the producer faces the row vector of input prices $\mathbf{v}^{\tau} \equiv [v_1^{\tau} v_2^{\tau} \ldots v_N^{\tau}]$ and minimises the cost of production by producing the observed level of output y^{τ} and using the observed input quantities represented by the column vector $\mathbf{x}^{\tau} \equiv [x_1^{\tau} x_2^{\tau} \ldots x_N^{\tau}]'$ (transposition is denoted by '), so that:[1]

$$\mathbf{v}^{\tau} \cdot \mathbf{x}^{\tau} = C^{\tau}(\mathbf{v}^{\tau}, y^{\tau}) \qquad [1]$$

or:

$$\mathbf{v}^{\tau} \cdot \mathbf{a}^{\tau} = \frac{C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})}{y^{\tau}} \equiv c^{\tau}(\mathbf{v}^{\tau}, y^{\tau}) \qquad [2]$$

where $C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$ is the minimum total cost function, $c^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$ is the minimum unit cost function, and $\mathbf{a}^{\tau} \equiv [a_1^{\tau} a_2^{\tau} \ldots a_N^{\tau}]'$ is the vector of input-output coefficients, which are defined as $a_i^{\tau} \equiv x_i^{\tau} / y^{\tau}$ for i = 1, ..., N.[2] In the remainder of this chapter, for simplicity of analysis, we assume that only one single output is produced.

A particular index number P, that is constructed by using only prices $(\mathbf{v}^0, \mathbf{v}^t)$ and quantities $(\mathbf{x}^0, \mathbf{x}^t)$ or $(\mathbf{a}^0, \mathbf{a}^t)$, is said to be "exact" for the aggregator function $c(\mathbf{v}^{\tau}, \bar{y})$, where $\bar{y}$ is a given reference level of production, if the following equality identically holds between the two observation points 0 and t:[3]

$$P = \frac{c(\mathbf{v}^t, \bar{y})}{c(\mathbf{v}^0, \bar{y})} \qquad [3]$$

An index is considered to be "exact" to an aggregator function in the sense that it gives the same numerical results that can be obtained from the ratio between two different levels of that function. It can be noted that if the technology is non-homothetic, changes in the reference production level $\bar{y}$ bring about non-proportional changes in input quantities and the input-output coefficients at given input prices. In other words, the aggregate input-price index number P is not invariant with respect to $\bar{y}$ when the input quantities are not non-homothetically affected by the level of production. The same applies to the level of technology.

The corresponding implicit index number of aggregate input quantities is given by:

$$\tilde{Q} \equiv \frac{\mathbf{v}^t \cdot \mathbf{x}^t}{\mathbf{v}^0 \cdot \mathbf{x}^0} / P \qquad [4]$$

The index of total cost of production is equal to the input price index multiplied by the input quantity index, that is:

$$\frac{p^t}{p^0} \cdot \frac{y^t}{y^0} = P \cdot \tilde{Q} \qquad [5]$$

where $p^\tau \equiv c^\tau(\mathbf{v}^\tau, y^\tau)$ is the unit cost of production at the observation point τ and, therefore, $p^\tau y^\tau = \mathbf{v}^\tau \cdot \mathbf{x}^\tau$. From the definition of the implicit index number of total factor productivity as $\tilde{\pi} \equiv \frac{y^t}{y^0} / \tilde{Q}$ and the equality [5], we have

$$\tilde{\pi} \equiv P / \frac{p^t}{p^0} \qquad [6]$$

It can also be seen that the index number of total factor productivity [6] is not invariant with respect to the reference level of production when the technological change is non-homothetic, except the special case of particular functional forms of the production or cost function. In general, in non-homothetic cases, there is no unique measure of total factor productivity change.

Fixed-weight index numbers

In the more specific case of a homothetic technology, it has been known at least since the work of Konüs and Byushgens (1926) in the field of measurement of the cost-of-living index, that some functional forms of index numbers are "exact" for particular functional forms of the aggregator functions of prices. The Laspeyres price index:

$$P_L \equiv \frac{\mathbf{v}^t \cdot \mathbf{a}^0}{\mathbf{v}^0 \cdot \mathbf{a}^0} \qquad [7]$$

and the Paasche price index:

$$P_P \equiv \frac{\mathbf{v}^t \cdot \mathbf{a}^t}{\mathbf{v}^0 \cdot \mathbf{a}^t} \qquad [8]$$

are exact for the linear functional form of the aggregator function $c(\mathbf{v}^\tau) \equiv \sum_{i=1}^{N} \overline{q}_i v_i^\tau$, where $\overline{q}_i = a_i^0$ in the case of the Laspeyres index and $\overline{q}_i = a_i^t$ in the case of the Paasche index. However, if the input-output coefficients change over time in a proportional way or if all the prices change in a fixed proportion,[4] the Laspeyres and Paasche indexes co-incide; otherwise neither of these two indices is an accurate measure of the aggregate input-price index.

Another index number, which can be regarded as a generalisation of the Jevons price index $P_J \equiv \Pi_{i=1}^{N}(v_i^t / v_i^0)^{1/N}$, is the Cobb-Douglas price index:

$$P_{CD} \equiv \Pi_{i=1}^{N}(v_i^t / v_i^0)^{\bar{s}_i} \qquad [9]$$

that is "exact" for the Cobb-Douglas aggregator function $c_{CD}(\mathbf{v}^\tau) \equiv \alpha\Pi_{i=1}^{N}(v_i^\tau)^{\bar{s}_i}$, where $\bar{s}_i \equiv \frac{v_i a_i}{\sum_{i=1}^{N} v_i a_i}$. If the input-cost shares $\bar{s}_i$ are not constant over time, this index number is not an appropriate measure of the aggregate input-price index.

Flexible-weight index numbers

By referring to more general functional forms of aggregator functions, Diewert (1976) discovered another class of "exact" index numbers, which he called "superlative" using a term that had been introduced by Fisher (1922, pp. 247-248) for an undefined notion of index number. The "superlative" index numbers are "exact" for polynomial functional forms that can be interpreted as second-order approximations to an unknown aggregator function that is twice differentiable. The Fisher ideal index number of input prices, which is defined as

$$P_F \equiv (P_L \cdot P_P)^{1/2} \qquad [10]$$

turns out to be "exact" for the aggregator function $c_F(\mathbf{v}^\tau) \equiv (\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij} v_i^\tau v_j^\tau)^{\frac{1}{2}}$ with $b_{ij} = b_{ji}$, in the sense that $c_F(\mathbf{v}^t)/c_F(\mathbf{v}^0) = P_F$. The Fisher index is invariant with respect to changes in the input-output coefficients that are induced by changes in relative prices according to the above-mentioned polynomial aggregator function.

Another "superlative" index number is that defined by Törnqvist (1936) as:

$$P_T \equiv \Pi_{i=1}^{N}(v_i^t / v_i^0)^{\frac{1}{2}(s_i^0 + s_i^t)} \qquad [11]$$

where $s_i^0 \equiv \frac{v_i^0 a_i^0}{\sum_{i=1}^{N} v_i^0 a_i^0}$ and $s_i^t \equiv \frac{v_i^t a_i^t}{\sum_{i=1}^{N} v_i^t a_i^t}$. This index number can be seen as a generalisation of the Cobb-Douglas index number and turns out to be "exact" for the translog aggregation function $c_T^\tau(\mathbf{v}^\tau) \equiv \exp(\alpha_0^\tau + \sum_{i=1}^{N} \alpha_i^\tau \ln v_i^\tau + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} \gamma_{jk} \ln v_i^\tau \ln v_k^\tau)$, where it can be noted that the zero-order parameter α_0^τ and the first-order parameters α_i^τ are assumed to vary, whereas the second-order parameters γ_{ij} are constant.[5] More precisely, since the aggregator function is technologically indexed, the Törnqvist index is "exact" for this aggregator function in a different way than the other indexes, *i.e.* it is identically equal to the geometric mean of two ratios: $P_T \equiv \Pi_{i=1}^{N}(v_i^t / v_i^0)^{\frac{1}{2}(s_i^0 + s_i^t)} = [(c_T^t(\mathbf{v}^t)/c_T^t(\mathbf{v}^0)) \cdot (c_T^t(\mathbf{v}^t)/c_T^t(\mathbf{v}^0))]^{1/2}$. The Törnqvist index number of input prices can also be constructed in the case of non-homothetic technological changes. Caves, Christensen and Diewert

(1982*b*, p. 1411), who found this result, stated that the Törnqvist index is superlative in a considerably more general sense than other known "superlative" index numbers. However, more recently, Diewert (1992*b*) discovered that the Fisher ideal index can also be "exact" for a particular technologically non-homothetic function. Moreover, while the Törnqvist index is exact for a geometric mean of two function ratios, the Fisher ideal index is identically equal to a single function ratio and, at the same time, it is the only known index number that satisfies certain important tests or desirable mathematical properties.

Fixed-weight versus flexible-weight index numbers

The fixed-weight index numbers of input prices that are defined by [7], [8] and [9] and the flexible-weight index numbers [10] and [(11] are all possible candidates for the approximation of the "true" unknown aggregating index number of input prices.[6] No index can be preferred *a priori* as the most accurate estimate of the "true" unknown index number. It is not possible to know if the input-substitution effects that are incorporated into the flexible-weight index numbers are, in fact, a disguised component of technological change, where input-substitutions are instead absent. In this latter case, a fixed-based index number could be more appropriate. On the other hand, if changes in relative prices determine a change in input-output coefficients that, in turn, bring about a significant change in cost shares, flexible-weight index numbers should, in principle, be preferred.

The index number of total factor productivity can be calculated in various ways by using formula [6] and substituting the index *P* with one of the available fixed-weight or flexible-weight index numbers of input prices. When the technological situation is not known, it is only possible to determine the extremes of the intervals of possible values of the "true" unknown index number. From the theory of cost-of-living index (see, for example, Fisher and Shell, 1972, Essay I; and Diewert, 1987), it can be established that $P_P \leq [(c(\mathbf{v}^t, y^0)/c(\mathbf{v}^0, y^0)]$ and $[(c(\mathbf{v}^t, y^t)/c(\mathbf{v}^0, y^t)] \leq P_L$, and if the function $c(\mathbf{v}^\tau, y^\tau)$ is homothetic with respect to the output level, then for any reference level of production $\bar{y}$, $P_P \leq [(c(\mathbf{v}^t, \bar{y})/c(\mathbf{v}^0, \bar{y})] \leq P_L$.[7]

Multilateral intertemporal and interspatial comparisons of productivity levels

Multilateral interspatial comparisons

Further problems arise when more than two regions or countries are simultaneously examined. Bilateral comparisons of productivity levels cannot be consistently chained since, in general, they are not transitive. For example, if we take three countries, α, β, and γ, the ratio of costs of production between α and γ can be assessed either directly or by linking together the cost ratio between α and β and that between β and γ as follows:[8]

$$\frac{p^\alpha}{p^\gamma} = \frac{p^\alpha}{p^\beta} \cdot \frac{p^\beta}{p^\gamma} \qquad [12]$$

This *circularity* or *transitivity condition* does not hold for the input-price and productivity components. The three output-price ratios are in fact decomposed as follows:

$$\frac{p^\alpha}{p^\gamma} = P^{\alpha\gamma} \cdot \frac{1}{\pi^{\alpha\gamma}}$$

$$\frac{p^{\alpha}}{p^{\beta}} = P^{\alpha\beta} \cdot \frac{1}{\pi^{\alpha\beta}} \quad [13]$$

$$\frac{p^{\beta}}{p^{\gamma}} = P^{\beta\gamma} \cdot \frac{1}{\pi^{\beta\gamma}}$$

where $P^{\alpha\gamma}$, $P^{\alpha\beta}$, $P^{\beta\gamma}$ are bilateral index numbers of input prices, and $\pi^{\alpha\gamma}$, $\pi^{\alpha\beta}$, $\pi^{\beta\gamma}$ are bilateral index numbers of productivity levels. These bilateral index numbers do not respect, in general, the following circularity or transitivity condition:

$$P^{\alpha\gamma} = P^{\alpha\beta} \cdot P^{\beta\gamma} \quad [14]$$

$$\pi^{\alpha\gamma} = \pi^{\alpha\beta} \cdot \pi^{\beta\gamma}$$

This can be verified by using the bilateral decomposition procedures that are based on one of the fixed-weight or flexible-weight index numbers mentioned above. A method for the multilateral comparison must be developed in order to permit the decomposition procedure to be invariant with respect to the order of pairs of countries or regions examined.

A possible method is parallel to that proposed by Fujikawa, Izumi and Milana (1995*a*) for a multilateral interspatial comparison of outputs, inputs and productivity, which was in turn derived from a method proposed by Diewert (1988). The basic idea is to account for the difference in output prices of each country and those of a hypothetical reference country. In the present context, the index number of the output cost of the examined country relative to a hypothetical reference country is defined as a weighted geometric average of bilateral output cost indices of the examined country with respect to the other countries:

$$\frac{p^{i}}{\overline{p}} = \left[\prod_{j=1}^{J}\left(\frac{p^{i}}{p^{j}}\right)^{\sigma_j}\right]^{\frac{1}{\sum_r \sigma_r}} \quad [15]$$

with σ_j denoting the appropriate weight for country *j*’s output cost (defined below). Taking into account equation [6], we have:

$$\frac{p^{i}}{\overline{p}} = \overline{P}^{i} \cdot \frac{1}{\overline{\pi}^{i}} \quad [16]$$

where:

$$\overline{P}^{i} \equiv \left[\prod_{j=1}^{J}\left(P^{ij}\right)^{\sigma_j}\right]^{\frac{1}{\sum_r \sigma_r}}$$

$$\frac{1}{\overline{\pi}^{i}} \equiv \left[\prod_{j=1}^{J}\left(\frac{1}{\pi^{ij}}\right)^{\sigma_j}\right]^{\frac{1}{\sum_r \sigma_r}}$$

The weighted geometric average of bilateral indices respects the important property of *country characteristicity:*[9] the characteristicity quantity weights of each pair of countries as a function of prices are used to construct the resulting multilateral index number.

Relative levels of the output price between country α and country γ can be decomposed as follows:

$$\frac{p^{\alpha}}{p^{\gamma}} = \frac{p^{\alpha} / \overline{p}}{p^{\gamma} / \overline{p}} = \frac{\overline{P}^{\alpha}}{\overline{P}^{\gamma}} \cdot \frac{\overline{\pi}^{\gamma}}{\overline{\pi}^{\alpha}} \qquad [17]$$

where $\overline{P}^{\alpha} / \overline{P}^{\gamma}$ is the multilateral index number of input prices between country α and country γ, whereas $\overline{\pi}^{\gamma} / \overline{\pi}^{\alpha}$ is the multilateral index number of productivity of country α relative to country γ. The weighted geometric average of all the bilateral indices permits us to overcome the non-transitive nature of particular bilateral indices, for example the Fisher ideal index:[10] it can be easily seen that, even if the bilateral indices P^{ij} are not transitive index numbers, the multilateral procedure that uses these indices as building blocks gives transitive results.

The definition of the methodology of multilateral interspatial comparisons is completed by specifying the weights σ_j. We note that:

i) If σ_j = 1/*J* (with *J* being the number of the examined countries) and P^{ij} are Fisher ideal input price indices, then [15]-[16] correspond to the so-called *EKS* method (Eltetö and Köves, 1964; Szulc, 1964); whereas if σ_j = 1/*J* and P^{ij} are Törnqvist input price indices, then [15]-[16] correspond to the so-called *CCD* method (Caves, Christensen and Diewert, 1982*a*). The weights defined by σ_j = 1/*J* have been named "democratic" weights by Diewert (1986, p. 27), since they are the same for every country.[11]

ii) If $\sigma_j = p^j y^j / (\sum_{r=1}^{J} p^r y^r)$, then [15]-[16] correspond to the "plutocratic" weight system suggested by Diewert (1986, p. 27).

iii) If $\sigma_j = y^j / (\sum_{r=1}^{J} y^r)$, then [15]-[16] correspond to the "own share" system suggested by Diewert (1986, p. 25).

Aggregation consistency over countries requires that the weights be based on the relative importance of the examined countries. If the weights are proportional to the production size of the countries, then the empirical results must be affected by a hypothetical or real splitting or aggregation of the countries. For this reason, the weighted systems *ii*) and *iii*) are preferable.[12] The "plutocratic" weight system *ii*), however, is not invariant to scale changes in the prices of any one country.

Finally, it should be emphasized that the methodology established here does not respect, in general, the so-called *additivity condition,* requiring that the country quantity levels that could be obtained by deflating their values at current prices by means of their own price indices sum up to the quantity level of the total group of countries. This condition has always guided the work of statisticians and accountants, but it is not necessarily valid from the viewpoint of economic theory. Aggregation over goods, which can differ not only in their intrinsic characteristics and quality, but also in their availability across time and space, is in fact additive only under very restrictive hypotheses. In general, it must be recognised that economic aggregates should not be constructed simply by adding up their constituent elements, but by adopting a correct aggregating procedure that, in general, is not linear.[13]

Multilateral intertemporal and interspatial comparisons

Multilateral comparisons of countries that are situated in two different periods of time could be consistently defined by referring to the hypothetical country in the base year. The index number of the output price of the examined country i at period t, relative to a hypothetical reference country in base period 0, is defined here as a weighted geometric average of bilateral comparisons of the output price of country i at period t and the output prices of all examined countries at base period 0:[14]

$$\frac{p^{it}}{\overline{P}^{0}} = \left[\prod_{j=1}^{J}\left(\frac{p^{it}}{p^{j0}}\right)^{\sigma_{j0}}\right]^{\frac{1}{\sum_r \sigma_{r0}}} \qquad [18]$$

with σ_{j0} being defined for country j's output cost in period 0.

Taking into account equation [6], we have:

$$\frac{p^{it}}{\overline{p}^{0}} = \left[\prod_{j=1}^{J}\left(P^{it,j0}\cdot\frac{1}{\pi^{it,j0}}\right)^{\sigma_{j0}}\right]^{\frac{1}{\sum_r \sigma_{r0}}} = \overline{P}^{it,0}\cdot\frac{1}{\overline{\pi}^{it,0}} \qquad [19]$$

where $P^{it,j0}$ is the aggregate index number of input prices of country i in period t with respect to country j at period 0, to be calculated by using one of the formulas discussed in the previous section, and:

$$\overline{P}^{it,0} \equiv \left[\prod_{j=1}^{J}\left(P^{it,j0}\right)^{\sigma_{j0}}\right]^{\frac{1}{\sum_r \sigma_{r0}}} \qquad [20]$$

$$\frac{1}{\overline{\pi}^{it,0}} \equiv \left[\prod_{j=1}^{J}\left(\frac{1}{\pi^{it,j0}}\right)^{\sigma_{j0}}\right]^{\frac{1}{\sum_r \sigma_{r0}}} \qquad [21]$$

The index number of the output price of the hypothetical country in the two periods of time t and 0 is given by:

$$\frac{\overline{p}^{t}}{\overline{p}^{0}} = \left[\prod_{j=1}^{J}\left(\frac{p^{it}}{\overline{p}^{0}}\right)^{\sigma_{it}}\right]^{\frac{1}{\sum_r \sigma_{rt}}} \qquad [22]$$

with σ_{jt} being defined as country j's output cost in period t.

Taking into account equation [6], we have:

$$\frac{\overline{p}^{t}}{p^{0}} = \overline{P}^{t,0} \cdot \frac{1}{\overline{\pi}^{t,0}} \qquad [23]$$

where:

$$\overline{P}^{t,0} \equiv \left[\prod_{i=1}^{J} \left(\overline{P}^{it,0}\right)^{\sigma_{it}}\right]^{\frac{1}{\sum_r \sigma_{rt}}}$$

$$\frac{1}{\overline{\pi}^{t,0}} \equiv \left[\prod_{i=1}^{J} \left(\frac{1}{\overline{\pi}^{it,0}}\right)^{\sigma_{it}}\right]^{\frac{1}{\sum_r \sigma_{rt}}}$$

Relative levels of the output price between country α in period t and country γ in period 0 can be decomposed as follows:

$$\frac{p^{\alpha t}}{p^{\gamma 0}} = \frac{p^{\alpha t} \;/\; \overline{p}^{0}}{p^{\gamma 0} \;/\; \overline{p}^{0}} = \frac{\overline{P}^{\alpha t,0}}{\overline{P}^{\gamma 0,0}} \cdot \frac{\overline{\pi}^{\gamma 0,0}}{\overline{\pi}^{\alpha t,0}} \qquad [24]$$

where ($\overline{P}^{\alpha t,0}$ / $\overline{P}^{\gamma 0,0}$) is the multilateral index number of input prices between country α in period t and country γ in period 0 (with $\overline{P}^{\alpha t,0}$ and $\overline{P}^{\gamma 0,0}$ defined as weighted averages of bilateral index numbers of input prices as in equation [20]), whereas ($\overline{\pi}^{\gamma 0,0}$ / $\overline{\pi}^{\alpha t,0}$) is the multilateral index number of productivity of country α in period t relative to that of country γ in period 0 (with $\overline{\pi}^{\gamma 0,0}$ and $\overline{\pi}^{\alpha t,0}$ defined as weighted averages of bilateral index numbers of productivity as in equation [21]).

Equation [24] can also be used for a multilateral intertemporal index number of productivity change for the same country between two periods of time, by replacing country γ in period 0 with country α in period 0. This multilateral intertemporal index number is, in general, different from the traditional bilateral index number, which is formulated by taking into account only the change in the examined country.

Direct and indirect effects of productivity change

Almost all empirical studies in the field of intertemporal and interspatial comparisons of productivity levels account for only the direct components. The accounting procedures take into account only the *direct* input costs and leave “unexplained” those indirect input costs that are incorporated into the direct intermediate inputs. This is a rather strong limitation of the analysis because technical progress and quality change are notoriously embodied in the goods and services used as inputs of production. Overlooking this component seriously limits the importance of the empirical conclusions, as intermediate inputs account for a relevant cost share – more than 60 per cent in many industries at the level of disaggregation of the analysis. In these cases, the indirect effects of productivity embodied in the intermediate inputs may be substantial.

When both direct and indirect effects of productivity are to be evaluated, the choice of the appropriate index numbers becomes even more important for the empirical results. This is due to the fact that the errors of approximation that are introduced by an inaccurate index number are amplified by the input-output relationships.

The incorporation into the input-output framework of an accounting procedure that is based on particular index numbers of input prices and productivity components brings about a revision of the Leontief inverse matrix. This matrix is used to evaluate total impacts on gross output, output prices and direct and indirect factor requirements. Some index numbers, as Laspeyres, Paasche and Cobb-Douglas indices, lead us to the familiar formulations of the input-output equations. Unitl recently, "superlative" index numbers were not incorporated into this framework.

We present below alternative versions of the methodology that was originally developed by Fujikawa, Izumi and Milana (1995*b*), who introduced the direct Törnqvist index numbers of input prices into the input-output accounting system. The two classes of fixed- and flexible-weight indices are reformulated as follows:

i) Fixed-weight aggregating index numbers of input prices:

$$\mathbf{P}_L \equiv \left[\mathbf{v}^t \cdot \hat{\mathbf{v}}^{0^{-1}} \cdot \mathbf{B}^0\right] \cdot \hat{\mathbf{p}}^{0,0^{-1}} \qquad [25]$$

where: $\mathbf{p}^{0,0} = \mathbf{v}^0 \cdot \hat{\mathbf{v}}^{0^{-1}} \cdot \mathbf{B}^0 = [1\ 1\ ...\ 1]$,

$$\mathbf{P}_P \equiv \left[\mathbf{v}^t \cdot \hat{\mathbf{v}}^{0^{-1}} \cdot \mathbf{B}^t\right] \cdot \hat{\mathbf{p}}^{0,t^{-1}} \qquad [26]$$

where: $\mathbf{p}^{0,t} = \mathbf{v}^0 \cdot \hat{\mathbf{v}}^{0^{-1}} \cdot \mathbf{B}^t$,

$$\mathbf{P}_{CD} \equiv \exp\left[\left(\ln \mathbf{v}^t - \ln \mathbf{v}^0\right).\left(\hat{\mathbf{v}}^0 \mathbf{B}^0 \hat{\mathbf{p}}^{0^{-1}}\right)\right] \qquad [27]$$

ii) Flexible-weight aggregating index numbers of input prices:

$$\mathbf{P}_F \equiv \left[\left(P_{L1}.P_{P1}\right)^{1/2} \ldots \left(P_{LN}.P_{PN}\right)^{1/2}\right] \qquad [28]$$

$$\mathbf{P}_T \equiv \exp\left[\left(\ln \mathbf{v}^t - \ln \mathbf{v}^0\right)\frac{1}{2}\left(\hat{\mathbf{v}}^0 \mathbf{B}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{v}}^t \mathbf{B}^t \hat{\mathbf{p}}^{t^{-1}}\right)\right] \qquad [29]$$

where $\mathbf{P}_L$, $\mathbf{P}_P$ and $\mathbf{P}_{CD}$ denote the vectors of sectoral Laspeyres, Paasche and Cobb-Douglas aggregating index numbers of input prices, $\mathbf{P}_F$ and $\mathbf{P}_T$ denote the vectors of sectoral Fisher and Törnqvist aggregating index numbers of input prices, the $\mathbf{B}^0$ and $\mathbf{B}^t$ are the input-output coefficient matrices, which are evaluated at the prices observed at base period 0, and therefore $\mathbf{v}^0$ = [1 1 ... 1] and $\mathbf{p}^0$ = [1 1 ... 1].

The matrices $\mathbf{B}^0$ and $\mathbf{B}^t$ can be defined in different alternative ways. If the direct input-output coefficients are represented by the two matrices $\mathbf{F}^\tau$ and $\mathbf{A}^\tau$, for τ = 0,*t*, $\mathbf{F}^\tau$ is the matrix of the input-output coefficients of primary factors, A^τ is the matrix of direct input-output coefficients of intermediate inputs, so that $\left(\mathbf{I}-\mathbf{A}^{\tau}\right)^{-1}$ is the Leontief inverse matrix defining the direct and indirect output requirements per unit of final outputs, two possible alternative formulations could be the following:

$$\mathbf{B}^{\tau} \equiv \mathbf{F}^{\tau}(\mathbf{I}-\mathbf{A}^{\tau})^{-1} \qquad [30]$$

which is associated to an input-price index number vector $\mathbf{v}^{\tau}$, defined as a vector of index numbers of primary factor prices $\mathbf{w}^{\tau}$, so that $\mathbf{v}^{\tau} \equiv \mathbf{w}^{\tau}$ or:

$$\mathbf{B}^{\tau} \equiv [\mathbf{A}^{\tau}\text{'}\,\mathbf{F}^{\tau}\text{'}]\text{'} \qquad [31]$$

which is associated to an input-price index number vector $\mathbf{v}^{r}$, defined as a vector of index numbers of intermediate input and primary factor prices, so that $\mathbf{v}^{\tau} \equiv [\mathbf{p}^{\tau} \quad \mathbf{w}^{\tau}]$. It should be noted that the vector $\mathbf{p}^{\tau}$ denotes both output and input prices, with net indirect taxes on intermediate inputs being considered here as primary factors.

Taking into account [6], the vectors of sectoral implicit index numbers of productivity are derived as follows:

iii) Fixed-weight index numbers of productivity:

$$\tilde{\pi}_L \equiv \mathbf{P}_P \cdot \left[\hat{\mathbf{p}}^{t} \cdot \hat{\mathbf{p}}^{0-1}\right]^{-1}$$

$$= \left[\mathbf{v}^{0} \cdot \hat{\mathbf{v}}^{0-1} \cdot \mathbf{B}^{0}\right] \cdot \hat{\mathbf{p}}^{0,t-1} \qquad [32]$$

$$\tilde{\pi}_P \equiv \mathbf{P}_L \cdot \left[\hat{\mathbf{p}}^{t} \cdot \hat{\mathbf{p}}^{0-1}\right]^{-1}$$

$$= \left[\mathbf{v}^{t} \cdot \hat{\mathbf{v}}^{0-1} \cdot \mathbf{B}^{0}\right] \cdot \hat{\mathbf{p}}^{t,t-1} \qquad [33]$$

where: $\mathbf{p}^{t,t} = \mathbf{v}^{t} \cdot \hat{\mathbf{v}}^{0-1} \cdot \mathbf{B}^{t}$,

$$\tilde{\pi}_{CD} \equiv \exp\left[\left(\ln \mathbf{v}^{t} - \ln \mathbf{v}^{0}\right).\left(\hat{\mathbf{v}}^{0}\mathbf{B}^{0}\hat{\mathbf{p}}^{0^{-1}}\right) - \left(\ln \mathbf{p}^{t} - \ln \mathbf{p}^{0}\right)\right] \qquad [34]$$

iv) Flexible-weight index numbers of productivity:

$$\tilde{\pi}_F \equiv \left[\left(\tilde{\pi}_{L1} . \tilde{\pi}_{P1} \right)^{1/2} \ldots \left(\tilde{\pi}_{LN} . \tilde{\pi}_{PN} \right)^{1/2} \right] \qquad [35]$$

$$\tilde{\pi}_T \equiv \exp\left[\left(\ln \mathbf{v}^t - \ln \mathbf{v}^0 \right) \frac{1}{2} \left(\hat{\mathbf{v}}^0 \mathbf{B}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{v}}^t \mathbf{B}^t \hat{\mathbf{p}}^{t^{-1}} \right) \right] - \exp\left(\ln \mathbf{p}^t - \ln \mathbf{p}^0 \right) \qquad [36]$$

Index numbers of productivity can be split into direct and indirect effect components, so that $\tilde{\pi}_{\cdot d\&i} = \tilde{\pi}_{\cdot d} \cdot \tilde{\pi}_{\cdot i}$, where $\tilde{\pi}_{\cdot d}$ represents the direct effect of productivity change and $\tilde{\pi}_{\cdot i}$ represents the indirect effect of productivity change that is incorporated into the intermediate inputs directly and indirectly required by the examined sectors. $\tilde{\pi}_{\cdot d}$ and $\tilde{\pi}_{\cdot i}$ can be computed by redefining the input-output coefficient matrix appropriately. To calculate $\tilde{\pi}_{\cdot d}$, the matrices $\mathbf{B}^0$ and $\mathbf{B}^t$ as defined in [31] are used, whereas the fixed-weight index numbers of total effects $\tilde{\pi}_{\cdot d\&i}$ and the Fisher ideal index number of total effects, the matrices $\mathbf{B}^0$ and $\mathbf{B}^t$ are those defined in [30]. For the Törnqvist index number of total effects, two alternative methods are possible: *i*) the first is based on the definition of the matrices $\mathbf{B}^0$ and $\mathbf{B}^t$ defined by [30]; *ii*) the second is based on the reduced form that can be obtained from the structural input-output system,[15] that is:

$$\tilde{\pi}_{T,d\&i} = \exp\left\{ \ln \tilde{\pi}_{T,d} . \left[\mathbf{I} - \frac{1}{2} \left(\hat{\mathbf{p}}^0 \mathbf{A}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{p}}^t \mathbf{A}^t \hat{\mathbf{p}}^{t^{-1}} \right) \right]^{-1} \right\} \qquad [37]$$

$$\tilde{\pi}_{T,d} = \exp\left[\left(\ln \mathbf{w}^t - \ln \mathbf{w}^0 \right) \frac{1}{2} \left(\hat{\mathbf{w}}^0 \mathbf{F}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{w}}^t \mathbf{F}^t \hat{\mathbf{p}}^{t^{-1}} \right) \right] + \exp\left[\left(\ln \mathbf{p}^t - \ln \mathbf{p}^0 \right) \frac{1}{2} \left(\hat{\mathbf{p}}^0 \mathbf{A}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{p}}^t \mathbf{A}^t \hat{\mathbf{p}}^{t^{-1}} \right) \right] - \exp\left(\ln \mathbf{p}^t - \ln \mathbf{p}^0 \right) \qquad [38]$$

$$\tilde{\pi}_{T,i} = \exp\left(\ln \tilde{\pi}_{T,d\&i} - \ln \tilde{\pi}_{T,d} \right) = \exp\left[\ln \tilde{\pi}_{T,d} \cdot \frac{1}{2} \left(\hat{\mathbf{p}}^0 \mathbf{A}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{p}}^t \mathbf{A}^t \hat{\mathbf{p}}^{t^{-1}} \right) . \left[\mathbf{I} - \frac{1}{2} \left(\hat{\mathbf{p}}^0 \mathbf{A}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{p}}^t \mathbf{A}^t \hat{\mathbf{p}}^{t^{-1}} \right) \right]^{-1} \right] \qquad [39]$$

The index number of total factor productivity defined by [38] is the traditional Törnqvist index of direct effects employed in international comparisons of productivity levels.[16] The Törnqvist indices of total, direct and indirect effects [37], [38] and [39] are employed by Fujikawa, Izumi and Milana (1995*b*).

Empirical evidence from recent empirical studies

A very limited number of studies have compared, under the same conditions of data information, differences in results that may be obtained by using alternative index numbers of productivity relative levels. As has been shown above, index numbers of productivity may yield different empirical results for the following main reasons: *i*) different functional forms of the index number; *ii*) different methodology used for multilateral comparisons; *iii*) different accounting framework and disaggregation of inputs; and *iv*) different production units examined (establishments, firms, industries, vertically integrated sectors).

Fixed-weight versus flexible-weight indices

It is well-known that, in the context of international comparisons, Laspeyres and Paasche indices can differ by more than 50 per cent (see, for example, Ruggles, 1967, pp. 189-90; and Kravis, Kenessey, Heston and Summers, 1975, p. 11). However, the divergence of the indices has to be assessed in each case since the price-induced effects can be determined by many factors . Further elements can be obtained by comparing the flexible-weight procedure that was used in a previous paper (Fujikawa, Izumi, and Milana, 1995*a*) with the Laspeyres-type and Paasche-type counterparts. In that paper, cost differences rather than cost ratios were decomposed into an input-price component and a technological component in a multilateral comparison of Japan, West Germany and the United States. The flexible-weight procedure is the so-called Bennet-type decomposition formula, which is parallel to the Törnqvist index number approach, in the sense that it takes natural values rather than logarithmic values of the examined variables (therefore, price differences rather than price ratios are accounted for).

As for the total technological component of sectoral cost differences, Charts 1-4 show that substantial differences can be observed in the results. In the Japan-US comparison, the largest differences in absolute terms can be noted in coal and coal products; wood products; construction; non-metallic products; whereas in relative terms the differences are more uniform across industries. With the exception of petroleum; electricity; tobacco; and finance, where the differences in results are relatively small, the Laspeyres-type technological component of cost differences is around 15 per cent higher than the Bennet-type counterpart. On the other hand, the Paasche-type measure of the same component is about 15 per cent lower. These results confirm that the two fixed-weight Laspeyres and Paasche formulas can differ widely (in this case by around 30 per cent) even at a high level of disaggregation.

In the West Germany-US comparison, the alternative measures of technological component of price differences appear to be less uniform than in the Japan-US comparison. However, large absolute differences can be noted in the following industries: coal and coal products; non-metallic products; paper and printing products; other manufacturing; communications; and all the service sectors excluding trade. The Laspeyres-type technological component of cost differences is around 20 per cent higher than the Bennet-type measure. The Paasche-type measure of the same component is about 20 per cent lower . The average distance between the two fixed-weight Laspeyres and Paasche indices is, therefore, around 40 per cent from the Bennet-type index.

Flexible-weight index numbers of direct and indirect effects of productivity change

To the best of our knowledge, the fourth type of the difference in results (total productivity *versus* direct productivity effects) has not been examined directly until recently. Fujikawa, Izumi and Milana (1995*b*) compare productivity levels in Japan relative to the United States in 1985, both at the industry and vertically integrated sector levels. The methodology used for the former was that established by Jorgenson *et al.*, which is based on the Törnqvist index number formula [38], whereas the methodology used for the latter was based on the integration of this index number into the input-output inverse matrix, as shown by formulas [37]-[39]. However, Fujikawa, Izumi and Milana (1995*b*) used the disaggregation of intermediate inputs given by the input-output tables, whereas

Jorgenson *et al.* used the aggregate of intermediate inputs within the KLEM (capital, labour, energy and material) production or cost specification. The latter method is followed at some cost: it not only imposes strong restrictions on the separability of sub-groups of inputs, but also prevents an integration of the index number methodology within the context of the input-output system in order to evaluate the direct and indirect effects of productivity relative levels.

In the specific case of the bilateral comparison of Japan and the United States in 1985, we examine the results obtained by Jorgenson and Kuroda (1992); Kuroda (1994); Denny *et al*. (1992); and Fujikawa, Izumi and Milana (1995*b*). At the industry level, relative productivity appears to be similar in these four studies, except in paper and wood products; electrical machinery; and precision instruments, where it appears to be much lower in Fujikawa, Izumi and Milana (1995*b*). In this study, relative productivity level in transportation equipment appears to be higher in Japan than in the United States, whereas it is much lower in the other studies. In general, total factor productivity was lower in Japan than in the United States in the majority of industries during 1985. Exceptions were chemical products; automobiles; transportation equipment; and communications.

In the study carried out by Fujikawa, Izumi and Milana (1995*b*), the breakdown of the intermediate input price component into primary input price and productivity components reveals that the productivity incorporated into the intermediate inputs used by the Japanese industries is much lower than in the United States. In no industry, at the level of disaggregation of the analysis, has this component turned out to be higher in Japan than in the United States. It is interesting to note that in some industries, such as chemical products; primary metals; transport equipment; and other industries n.e.c., the direct productivity component is higher in Japan than in the United States, but is completely offset by the lower indirect productivity component. Moreover, in some of these sectors, such as primary metals and transport equipment, the indirect productivity component more than offsets the direct productivity component, thus leading to an unfavourable total effect on productivity levels. In these cases, the vertically integrated sector presents an overall productivity comparison, which has the opposite sign to the direct productivity component that is superficially observed at the industry level. Technological gaps between the countries could appear substantially different if the analysis is carried out at the level of the whole vertically integrated sector rather at the industry level.

Conclusion

Relative productivity levels, both in intertemporal and interspatial comparisons, are difficult to measure. In this chapter we have shown that a number of problems arise and, in general, cannot be solved in a simple way. The first of these problems can be associated with the familiar aggregation problem in the case of multiple inputs and multiple outputs. Since relative productivity levels are defined as ratios between aggregating index numbers of outputs and aggregating index numbers of inputs, appropriate estimates of these index numbers must be found.

From the theoretical literature, it is well-known that only under very restrictive conditions do the aggregates of outputs and inputs exist. In these particular conditions, however, the “true” aggregating index number remains unknown, and the best we can do is to find the extremes of possible values of this index number. Among the known functional forms of the index numbers, there are advantages and disadvantages in using each of the alternative estimating functional forms, since each may contain errors of specification in a direction that is also unknown. A possible strategy is to calculate alternative appropriate index numbers and to evaluate the divergence of the results. Flexible-weight index numbers usually give results that lie between those obtained by the fixed-weight Laspeyres and Paasche index numbers. Our empirical findings show that these two indices may differ widely and may change substantially the picture of the phenomenon under examination.

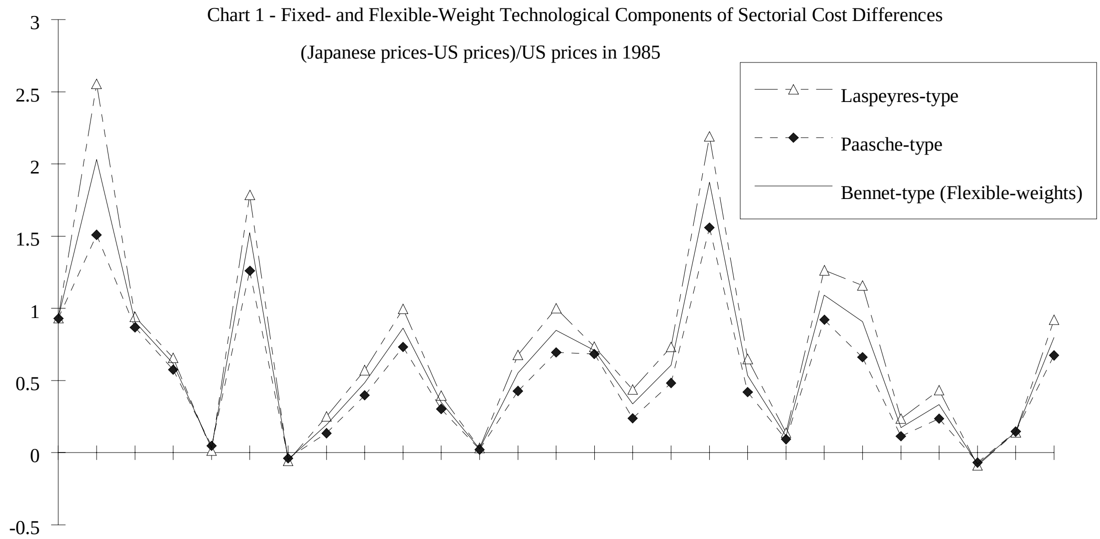


Chart 1 - Fixed- and Flexible-Weight Technological Components of Sectorial Cost Differences
(Japanese prices-US prices)/US prices in 1985
Laspeyres-type
Paasche-type
Bennet-type (Flexible-weights)
3
2.5
2
1.5
1
0.5
0
-0.5

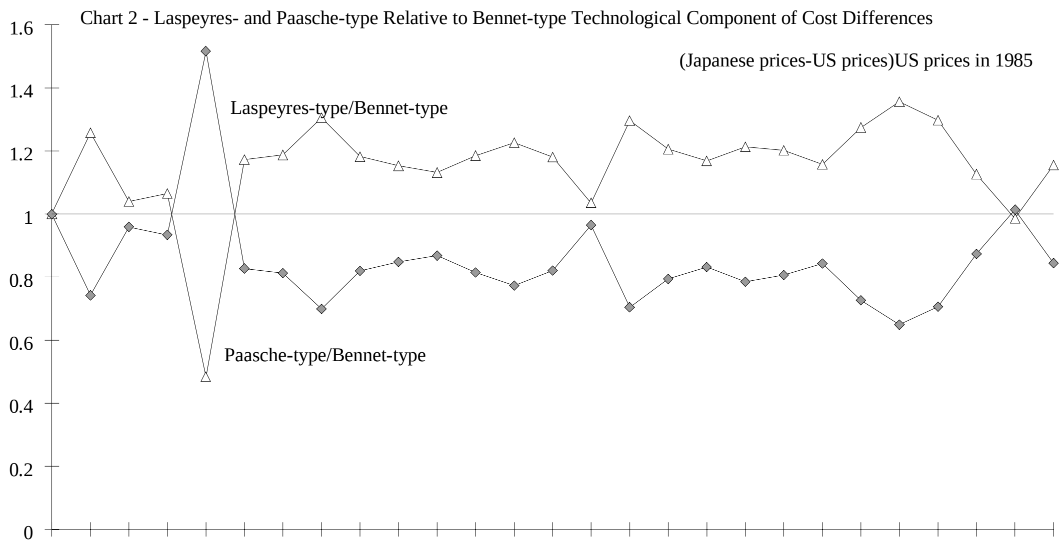


Chart 2 - Laspeyres- and Paasche-type Relative to Bennet-type Technological Component of Cost Differences
(Japanese prices-US prices)US prices in 1985
Laspeyres-type/Bennet-type
Paasche-type/Bennet-type
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0

Chart 3 - Fixed- and Flexible-Weight Technological Components of Sectorial Cost Differences

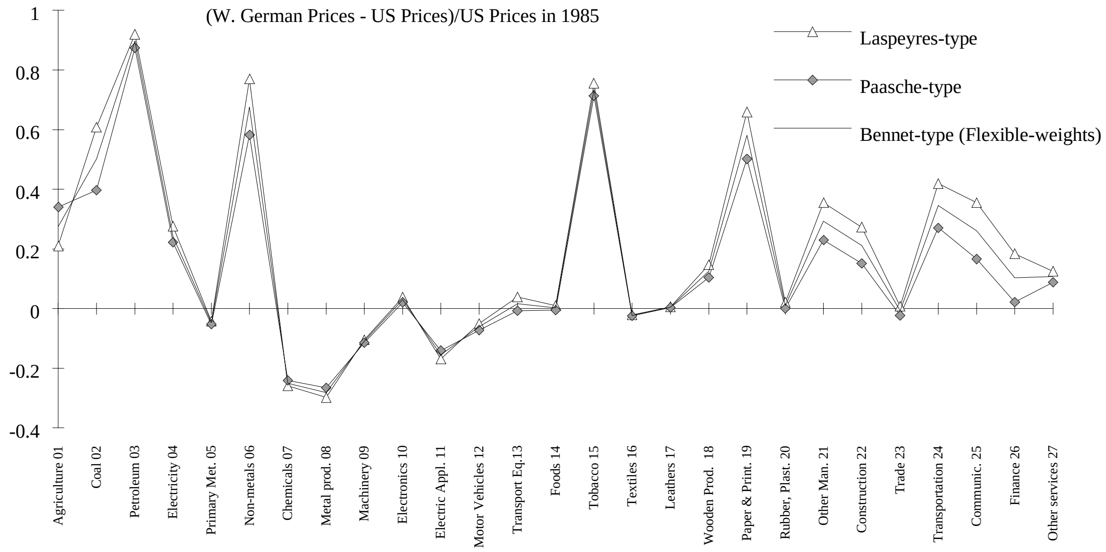


Chart 4 - Laspeyres- and Paasche-type Relative to Bennet-type Technological Component of Cost Differences

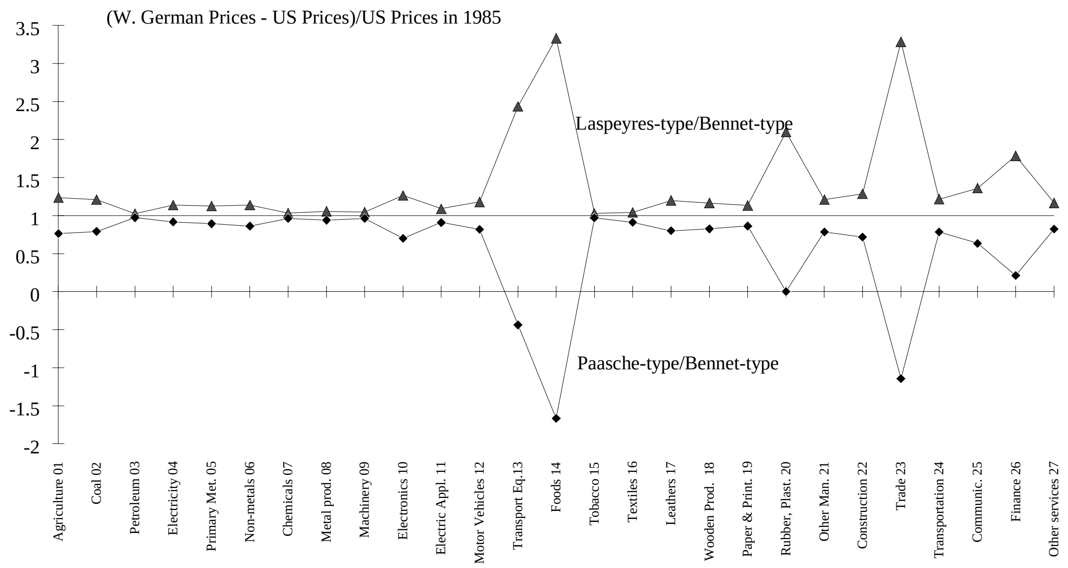


Chart 5 Relative Direct and Indirect Productivity in Japan, 1985 (U.S. = 1.00)

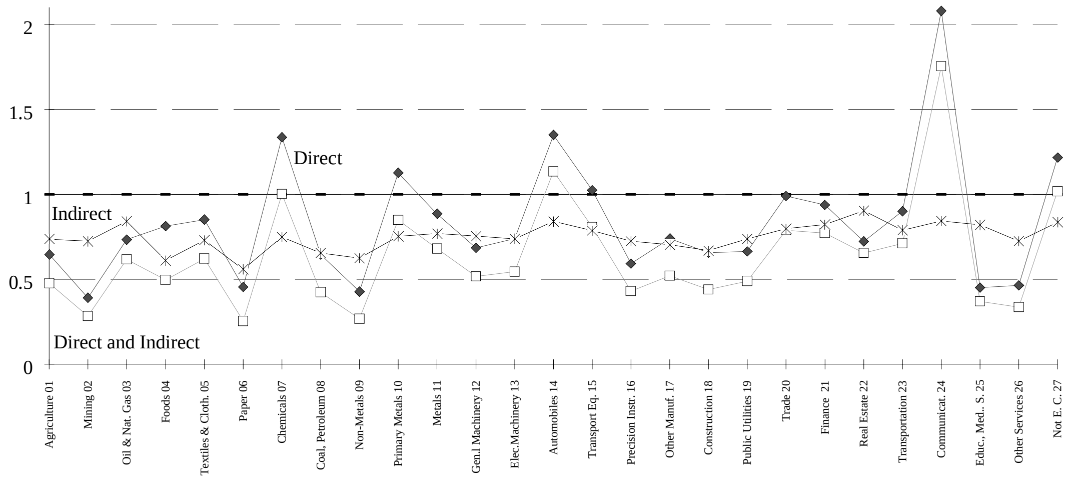
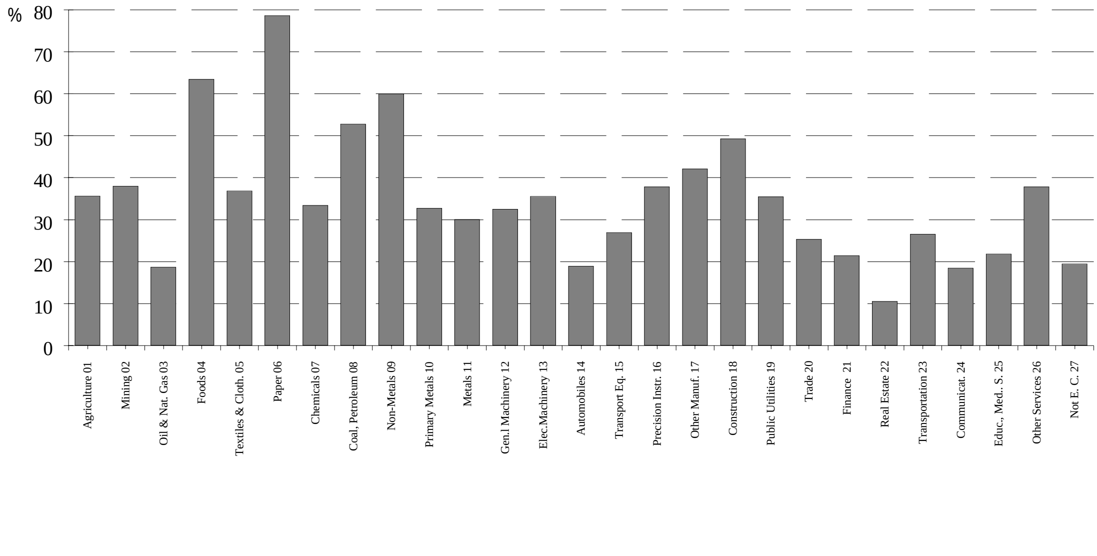


Chart 6 Difference in Measurement of Relative Productivity Levels between Japan and the U.S. in 1985 (Percentage Difference of Productivity between Industries and Vertically Integrated Sectors)



Equally important are the problems of making index numbers invariant with respect to the order of bilateral comparisons in a multilateral context. To solve this problem, a methodology has been established, which is consistent with both intertemporal and interspatial comparisons. Each of the alternative functional forms can be used within the outlined methodology. However, the chosen solution does not give a definite empirical answer, since the results are not invariant if a new country is introduced into the analysis.

The index number approach has also been incorporated into the framework of input-output analysis to take account of indirect effects of relative productivity levels that are incorporated in the intermediate inputs. Using an input-output inverse matrix, which is redefined to be fully consistent with the specific functional form of the chosen index number, this approach permits us to calculate the traditional direct as well as the indirect relative productivity levels. The empirical findings in the case of a bilateral comparison between Japan and the United States show surprising results. Not only are the indirect effects of productivity significant, but they very often turn out to be of opposite sign with respect to the direct effects, and are sometimes higher than the latter . The relative productivity gap between Japan and the United States in 1985 turns out to be much higher if we consider the whole economic system behind the production of the single good (the so-called vertically integrated sector) rather than that observable at the industry level.

NOTES

1. Price and quantity vectors are, respectively, defined as row and column vectors. Therefore, transposition signs are not needed when writing inner products.
2. By Shephard's lemma $\mathbf{x}^{\tau} = \nabla_{\mathbf{v}} C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$ and, since $C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$ is a linear homogeneous function of $\mathbf{v}^{\tau}$, by Euler's theorem $C^{\tau}(\mathbf{v}^{\tau}, y^{\tau}) = \mathbf{v}^{\tau} \cdot \nabla_{\mathbf{v}} C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$, hence $\mathbf{v}^{\tau} \cdot \mathbf{x}^{\tau} = C^{\tau}(\mathbf{v}^{\tau}, y^{\tau})$.
3. A particular aggregator function of input prices $c(\mathbf{v}^{\tau}, \bar{y})$ differs from the unit cost function $c^{\tau}(\mathbf{v}^{\tau}, \bar{y})$ because it does not take into account the effects of technical change and changes in the output level and therefore is not technologically indexed. In the general case, we can establish a given reference technology and a given output level within the unit cost function to determine a particular aggregator function, which however can change as the reference technology and the level of output are changed (see, for example, Fuss and Waverman, 1992, pp. 64-79).
4. These two cases correspond, respectively, to the well-known basic hypotheses of the Leontief and Hicks aggregation procedures.
5. Denny and Fuss (1983) pointed out that the econometric estimation approach can be dealt with more general translog functions, where also the coefficients on the second-order terms in input prices can be assumed to be different between the two situations under comparison.
6. Both fixed- and flexible-weight index numbers include other functional forms that might be taken into consideration. However, the index numbers that are defined by [7]-[11] are the best-known and most widely used.
7. The theory of the input price indices is largely isomorphic to that of the cost-of-living index. In the general non-homothetic case, $P_L \geq P_P$ or $P_L < P_P$, but it is always possible to define one particular reference output level y^* that is a weighted average of y^0 and y^t such that $P_L \geq [c(\mathbf{v}^t, y^*) / c(\mathbf{v}^0, y^*)] \geq P_P$ if $P_L \geq P_P$ or $P_L \leq [c(\mathbf{v}^t, y^*) / c(\mathbf{v}^0, y^*)] \leq P_P$ if $P_L \leq P_P$. In other words, we can define both extremes of the range of possible values of the "true" unknown index number that corresponds to the particular output level y^*. In the non-homothetic case, however, at other output levels, the "true" unknown index number has different values and we can only know one extreme or the other of the range of its possible values (see, for example, Muellbauer, 1972; Blackorby, Schwarm and Fisher, 1986; Diewert, 1987; Fisher, 1988).
8. We do not discuss here the methodology of measuring cost ratios for a given product between two countries. Alternative methods include: the Purchasing Power Parity approach and the direct measure of input prices and/or input quantities (see, for example, Pilat, 1991; van Ark and Pilat, 1993, 1994; van Ark, 1993).
9. This term is due to Drechsler (1973).
10. Kravis and Lipsey (1991) pointed out that "Fisher indexes are not transitive and few would favor them for multilateral comparisons". Transitivity is respected here even if the P^{ij} s are Fisher ideal indices.
11. See van Yzeren (1988, pp. 163-64) and Diewert (1996) for other alternative weighting systems.
12. Diewert (1988, p. 84, fn. 2) cites van Yzeren (1983) and Hill (1984, p. 131) among the first informal discussions of this consistency in country aggregation property.
13. This, however, may not be convenient for practical purposes. Hill (1988, p. 145), discussing chain indices, claimed: "They do not respect the rules of arithmetic or accounting constraints. The fact that chain indices are not additively consistent is a distinct disadvantage when working with a set of interdependent variables within an overall accounting framework or macroeconomic model, and this is another reason why fixed weight indices remain popular".
14. The first multilateral interspatial *and* intertemporal comparisons were made by Denny and Fuss (1980), Denny, Fuss and May (1981), who used a particular region in the base year as the reference point for

comparisons, and Caves, Christensen and Tretheway (1981), who used a procedure that is similar to the *CCD* method.

15. The structural input-output system from which [37], [38] and [39] are derived is given by:

$$(\ln \mathbf{p}^t - \ln \mathbf{p}^0) = (\ln \mathbf{p}^t - \ln \mathbf{p}^0)\frac{1}{2}(\hat{\mathbf{p}}^0 \mathbf{A}^0 \hat{\mathbf{p}}^{0^{-1}} + \hat{\mathbf{p}}^t \mathbf{A}^t \hat{\mathbf{p}}^{t^{-1}})$$

$$+(\ln \mathbf{w}^t - \ln \mathbf{w}^0) \cdot \frac{1}{2}(\hat{\mathbf{w}}^0 \mathbf{F}^0 \hat{\mathbf{w}}^{0^{-1}} + \hat{\mathbf{w}}^t \mathbf{F}^t \hat{\mathbf{w}}^{t^{-1}}) - \ln \tilde{\pi}_{T,d} \ ,$$

where $\tilde{\pi}_{T,d}$ can be interpreted as the implicit Törnqvist index number of direct technological component of the price index numbers $\mathbf{p}^t / \mathbf{p}^0$ (see Fujikawa, Izumi and Milana, 1995*b*, pp. 8-9, for an extensive discussion of this methodology).

16. See, for example, Jorgenson and Nishimizu (1978, 1981); Jorgenson, Kuroda and Nishimizu (1987); Jorgenson and Kuroda (1990, 1992); Denny *at al.* (1992); and Fuss and Waverman (1992). All these empirical studies, however, employed aggregated intermediate inputs.

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