A NET PROFIT APPROACH TO PRODUCTIVITY MEASUREMENT, WITH AN APPLICATION TO ITALY

by

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Abstract

We develop an approach to productivity measurement based on profit functions. We do this within a framework where non-separable outputs and inputs can be aggregated together, rather than separately as with conventional productivity indexes defined as ratios of output to input indexes. Our approach permits us to construct *net* aggregates that are always linearly homogeneous even in the non-homothetic production case. We apply our approach to data for Italian industries for 1970-2003 from the *EU KLEMS* project. Homotheticity seems to be the exception rather than the rule during the period of 1970-2003 in Italy. We find that the negative trend of productivity noted recently in this country almost disappears with the proposed measure. Severe limitations still remain in this exercise, including the assumption that producers are price takers in both input and output markets and a lack of correction for short-run cyclical behaviour.

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1. Introduction

> “[…] *Although most attention in the literature is devoted to price indexes, when you analize the use to which price indexes are generally put, you realize that quantity indexes are actually most important. Once somehow estimated, price indexes are in fact used, if at all, primarily to ‘deflate’ nominal or monetary totals in order to arrive at estimates of underlying ‘real magnitudes’ (which is to say,* quantity *indexes!)*”.
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> P.A. Samuelson and S. Swamy (1974, pp. 567-568)

Economic growth is the main hope for more jobs, tax revenue for government coffers without higher tax rates, and international bargaining power. The nominal value of the output of a nation can change because of changes in output versus input prices, changes in the amounts used of input factors, technical change, and returns to scale effects. There is interest in the decomposition of economic growth because the different sources have very different determinants. This paper focuses on the measurement of technical change taking into account the returns to scale effects.

In the currently conventional approach to productivity measurement based on index numbers, inputs of production are often assumed to be strongly separable from outputs (implying constant returns to scale) and changes in production technology (implying Hicks neutral technical progress)[2]. In this conventional approach, total factor productivity growth (TFPG) index can be interpreted also as a measure of technical change. But when non-constant returns to scale are not ruled out, then TFPG includes both technical change and returns to scale components.[3]

Recent studies allow for scale economies and correct index numbers for the degree of increasing returns to scale derived from an “external” source of information as, for example, parametric estimations, but in general the assumption of weak input-output separability is maintained. Here, we instead use the profit function as the framework for devising productivity

[2] It may seem inappropriate to distinguish between strong and weak separability when there are only two groups of commodities. However, as it is shown in Appendix A, we can distinguish two special cases of homothetic separability between outputs and inputs: the case of constant returns to scale, in which the input-output ratios are not affected by a third (hidden) factor represented by the internal or external scale of production, and the case of nonconstant returns to scale, in which the input-output ratios are affected by this factor. By introducing a third argument, we can formally speak of strong and weak input-output separability.

[3] The methods for making comparative indexes differ depending on whether the comparison involves just two production scenarios or multiple ones. Most of index number theory focuses on the bi-lateral comparisons case, and we do so here as well. Here we refer to indexes for bilateral comparisons of productivity, ideally taking account of all purchased factors of production, as total factor productivity growth indexes (TFPG), while keeping in mind that these indexes can also be used for bi-lateral comparisons among production units by making the notational substitution of symbols used to designate the production units for those used to designate the two time periods. For a recent survey of index number methods for measuring the productivity of nations, see Diewert and Nakamura (2006).

measures, building on the seminal research of Diewert and Morrison (1986) and Kohli (1990), which was based on the use of the revenue function[4].

Here we extend the Diewert-Kohli-Morrison framework to the case of a general input-output relationship where the returns to scale may be non-constant and also the induced changes in the structure of production are not necessarily homothetic. Therefore, input-output separability is not imposed *a priori.* Under this condition, inputs and outputs cannot be aggregated separately, so productivity cannot be measured as in the conventional approach. Instead an appropriate scalar value is defined summarizing the observed changes in the so-called "netput" vector relative to the scale of output, and corresponding to relative change in output due to technical change. In this general framework, the analysis is carried out using indicators which allow for unrestricted price-induced input-output substitution effects.

We construct indicators of "netput" price and quantity changes that satisfy desired properties for aggregation even in the presence of non-constant returns to scale.[5] The resulting normalized aggregate index is net of the effects of returns to scale; it represents technical change. The returns-to-scale effects are, instead, incorporated into the general price-induced substitutions and are taken into account implicitly by the economic index number formula. This is accomplished by defining indexes in the spaces of input-output quantity and price transformation functions that are always homogeneous by construction with respect to their arguments.

The rest of the paper is organized as follows. The second section develops profit-based productivity indicators that do not aggregate inputs and outputs separately and are consistent with non-homothetic effects of non-constant returns to scale. The third section develops the net profit based productivity measurement and demonstrates that, in the special case of constant returns to scale, the proposed indicator is equivalent to the traditional relative change of TFPG. The fourth section presents an application to the case of Italy using the database of the *EU KLEMS* project, and reveals that the decline in productivity noted recently in this country almost disappears with the proposed measure of productivity change. The fifth section concludes. Appendix A reviews separability conditions for outputs, inputs, and technical change that are defined within a general description of the production technology. Appendix B reviews some of the desired properties for index numbers. Appendix C examines the special importance of homogeneity in aggregation and the development of productivity indicators.

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[4] See also Morrison and Diewert (1990a, 1990b) and Morrison (1999).

[5] In constructing indicators of "netput" price and quantity change, and using these to form productivity indicators, we are also building on the work of Sono (1945, 1961), Leontief (1947a, 1947b), Samuelson (1947, 1950, 1953), Debreu (1951), Farrell (1957), Uzawa (1964), McFadden (1966), Diewert (1971, 1973, 1976, 1998, 2000, 2005), Samuelson and Swamy (1974), and Swamy (1985), Balk (1998), and especially Diewert and Morrison (1986) and Kohli (1990).

2. A Net Profit Function Framework

"The profit function takes the high ground; it is the most sophisticated representation of the technology"

(R. Färe and Primont, 1995, p. 149)

2.1 The producer's maximization problem

Following Samuelson (1950, p. 23), Debreu (1959, p. 38), and Diewert (1973), we can interpret some of the inputs as negative outputs and include them in **y** rather than in **x**. The output aggregating function $g^t(\mathbf{y})$, if it exists, can be interpreted as a net-quantity aggregator. If all the intermediate inputs used in production are considered as negative outputs (so that the vector **x** represents only the inputs of primary factor services), then the net output quantity aggregator $g^t(\mathbf{y})$ has the meaning of a *real value-added function*.

The conventional approach to productivity measurement assumes input-output separability. If technical change causes non-homothetic shifts in the space of the input quantities for given output levels, then the transformation function would be indexed to the technology in the parameters involved (or in its functional form) and the effects of this change could not, in general, be isolated. Thus with Hicks-neutral technical change, the function $F^t(\mathbf{x})$ can be indexed to technical change and can be written as $A^t \cdot F(\mathbf{x})$, where A^t is a separable technical change variable and the function F(·) is not subject to change. However, real world technical change could conceivably affect the whole internal structure of the functional relationship.

We follow an approach that is based on the duality between the production possibility frontier and cost, revenue, and profit functions along the lines of the pioneering contributions of Uzawa (1964), McFadden (1966), and Diewert (1971, 1973, 1974).[6]

The value function of the (static) profit maximization problem for a production unit operating in the long-run equilibrium is given by:

(2-1) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \max_{\mathbf{y},\mathbf{x}} \left\{ \mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x} : (\mathbf{x},\mathbf{y}) \in S^t \right\}$$

where $\mathbf{p} = [p_1, p_2, \ldots p_M]$ is a row vector of M output prices, $\mathbf{w} = [w_1, w_2, \ldots w_N]$ is a row vector of N input prices, and $\Pi^t(\mathbf{p},\mathbf{w})$ is the long-run equilibrium profit function at time t. There is no loss of generality if the levels of all output and input quantities are normalized by one elementary quantity, say the i^{th} output, which is then treated as representative of the scale of the whole production activity. All the solutions for quantities are thus stated in relative terms rather than in

[6] We deliberately shall not make explicit use of the concept of distance functions since we want to remain in the realm of economically defined optimal cost, revenue and profit functions. However, each of these value functions can be interpreted as a distance function in the space of their argument variables.

absolute units of measure. Long-run general-equilibrium market forces are assumed to determine the scale of production activities.[7]

There is a dual relationship between the profit function $\Pi^t(\mathbf{p},\mathbf{w})$ and the transformation function $T^t(\mathbf{y},\mathbf{x})$=0, which is the contour of the production possibility set S^t.

The profit function completely characterizes the technology of a production unit, in the sense that it contains all the information needed to describe the production possibility frontier $T^t(\mathbf{y},\mathbf{x})=0$.[8] $T^t(\mathbf{y},\mathbf{x})$ and $\Pi^t(\mathbf{p},\mathbf{w})$ are, respectively, almost homogeneous and linearly homogeneous in their arguments (see, for example, Lau, 1972).

The maximization problem that defines the profit function in (2-1) can be decomposed into different stages. Suppose that the optimized cost and conditional revenue functions are as follows:

(2-3) $$\mathrm{C}^t(\mathbf{w},\mathbf{y}) \equiv \min_{\mathbf{x}} \{\mathbf{w}\cdot\mathbf{x} : (\mathbf{y},\mathbf{x}) \in S^t\} \text{ and}$$

(2-4) $$\mathrm{R}^t(\mathbf{p},\mathbf{x}) \equiv \max_{\mathbf{y}} \{\mathbf{p}\cdot\mathbf{y} : (\mathbf{y},\mathbf{x}) \in S^t\}.$$

A simultaneous optimal solution leads us to the long-run equilibrium profit function defined by (2-1):

(2-5) $$\begin{aligned}\Pi^t(\mathbf{p},\mathbf{w}) &\equiv \max_{\mathbf{y},\mathbf{x}} \{\mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x} : (\mathbf{x},\mathbf{y}) \in S^t\} \\ &= \max_{\mathbf{y}} \{\mathbf{p}\cdot\mathbf{y} - \mathrm{C}^t(\mathbf{w},\mathbf{y})\} && \text{using (2-3)} \\ &= \max_{\mathbf{x},} \{\mathrm{R}(\mathbf{p},\mathbf{x}) - \mathbf{w}\cdot\mathbf{x}\} && \text{using (2-4)}\end{aligned}$$

In long-run equilibrium, Lau (1972, p. 284) has shown that with the degree $k<1$ of decreasing returns to scale in outputs

(2-6) $$\Pi^t(\mathbf{p},\mathbf{w}) = (1-k)\,\mathrm{R}[\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})] \text{ and } \mathrm{C}[\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w})] = k\,\mathrm{R}[\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})] \quad \text{using (2-5),}$$

and with the degree $k=1$ of constant returns to scale $\Pi^t(\mathbf{p},\mathbf{w}) = \mathrm{R}(\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})) - \mathrm{C}(\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w}))$ once the indeterminacy problem of the absolute levels of the variables has been solved. (Recall that the partial-equilibrium problem of profit maximization is indeterminate if $k=1$ and is not workable if $k>1$.) The levels of $\mathbf{x}(\mathbf{p},\mathbf{w})$ and $\mathbf{y}(\mathbf{p},\mathbf{w})$ are both consistent with the optimization problem (2-5).

Whether the cost-minimizing, revenue-maximizing, or "long-run equilibrium" profit maximizing solutions are to be considered as the closest to the producer's behavior at the particular time *t* depends on the specific conditions of the case examined. In general, the (conditional) short-run revenue and cost functions exhibit, in the short run, *decreasing* returns to

[7] With constant returns to scale, an *indeterminacy problem* arises in the (long-run) partial equilibrium context. The profit maximization problem may be solved in terms of absolute levels of the variables only in the general equilibrium of the economy.
[8] This sort of relationship is also known as a transformation function. As McFadden (1978, p. 92) has clarified, at its given value, the profit function is itself a *price possibility frontier* or transformation function defined in the space of producer's output and input prices.

scale in the *variable* outputs and/or inputs at given *fixed* reference levels of other outputs and/or inputs. Also in cases where the economies of scale are constant or even increasing all outputs and inputs are allowed to vary.

To encompass all these solutions in a unified approach, McFadden (1978, p. 66) has proposed the following *general restricted profit function* that includes, as particular cases, the cost (with negative sign), revenue and profit functions:

$$\text{(2-7)} \qquad \Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \max_{\mathbf{z}}\left\{\mathbf{q}\cdot\mathbf{z} : (\mathbf{z},\mathbf{k}) \in S^t\right\}$$

where $\mathbf{q}$ is a (row) vector of prices of *variable* outputs and/or (negative) variable inputs, the quantities of these outputs and (negative) inputs are the elements of a (column) vector $\mathbf{z}$, and $\mathbf{k}$ is a (column) vector of quantities of *fixed* outputs and/or (negative) inputs. McFadden (1978, p. 61) called a vector of (positive) outputs and (negative) inputs a vector of *net outputs* or *netputs.*[9]

By appropriately defining the *variable* output and/or input quantities as components of $\mathbf{z}$, the *conditional* output and/or input quantities as components of $\mathbf{k}$, and the prices of the elements of $\mathbf{z}$ as components of $\mathbf{q}$, the cost, revenue and long-run equilibrium profit functions can be obtained from McFadden's restricted profit function defined by (2-7) as follows

(*I*) If $\mathbf{z} \equiv -\mathbf{x}$, $\mathbf{q} \equiv \mathbf{w}$, and $\mathbf{k} \equiv \mathbf{y}$, then the McFadden general restricted profit function represents a (negative) minimum cost function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv -C^t(\mathbf{w},\mathbf{y})$ (using (2-3) and noting that the definition of the cost function assigns a positive algebraic sign to inputs and a negative sign to the outputs that are treated as inputs);

(*II*) If $\mathbf{z} \equiv [\mathbf{y}\ -\mathbf{x}_1]'$, $\mathbf{q} \equiv [\mathbf{p}\ \mathbf{w}_1]$, and $\mathbf{k} \equiv -\bar{\mathbf{x}}_2$, with the input quantities being partitioned into *variable* inputs $\mathbf{x}_1$ and *fixed* inputs $\bar{\mathbf{x}}_2$, and their prices $\mathbf{w}$ being partitioned accordingly, then McFadden's general restricted profit function represents a revenue function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv R^t([\mathbf{p} \vdots \mathbf{w}_1], \bar{\mathbf{x}}_2)$ (using (2-4)); and

(*III*) If $\mathbf{z} \equiv [\mathbf{y}\ (-\mathbf{x})]'$ and $\mathbf{k} = \mathbf{0}$, then McFadden's general restricted profit function *ceases to be restricted* and represents a long-run equilibrium profit function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^t(\mathbf{p},\mathbf{w})$ (using (2-5)).

McFadden's profit function refers to *net* profits only in the unrestricted case obtained with the translations (*III*).[10] This is the case we focus on in this paper.

[9] Duality between the producer's transformation function in the space of quantities and the cost, revenue and profit functions in the space of prices can be studied in a compact form using McFadden's general restricted profit function.

[10] In the other cases, this form defines optimal values that are *not* net profits. The translations (*I*) and (*II*) lead us to cost and gross revenue functions, respectively.

2.2 The general conditional net-profit function

We can separate the long-run equilibrium profit maximization problem into a two- stage profit maximization procedure. We base this theory on the basic contributions of Lau (1972) and Diewert (1974). In describing the theory of profit functions of technologies with multiple inputs and outputs, Lau (1972) considered the unrestricted case of long-run equilibrium, but he suggested that “fixed inputs and outputs may be incorporated easily in this framework” (p. 281, fn. 3). In line with Lau’s (1972) developments, we proceed now to introduce two conditional net profit functions according two different perspectives.

The first perspective corresponds to the Marshallian view of the producer’s partial-equilibrium problem, which consists in maximizing net profits by choosing the optimal levels of the outputs, with no explicit attention for cost-minimizing adjustments in the techniques of production. In this context, the (conditional) profit function is defined at given level of $\mathbf{x}$ on the production frontier

(2-8) $$\Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x}) = \mathrm{R}^{t}(\mathbf{p},\mathbf{x}) - \mathbf{w}\cdot\mathbf{x}$$

where $\Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x})$ can be defined as the partial-equilibrium “Marshallian” net profit function, which is conditional to the vector of inputs $\mathbf{x}$.

The “Marshallian” net profit function (2-8) may have a value between those of the short-run “Marshallian” net-profit function” (where the inputs remain fixed at the level $\overline{\mathbf{x}}$) and the long-run equilibrium profit function, that is

(2-9) $$\Pi^{M\,t}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}}) \le \Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x}) \le \Pi^{t}(\mathbf{p},\mathbf{w})$$

The difference between $\Pi^{M\,t}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}})$ and $\Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x})$ lay in the fact that, although $\overline{\mathbf{x}}$ and $\mathbf{x}$ are both assumed to be on the production frontier, the former is fixed and remains unchanged, whereas the latter is quasi-fixed and is allowed to change towards its optimal level.

The second perspective stems from the Paretian theory of the firm, which is to maximize profits through cost-minimizing adjustments in optimal input combinations at given levels of outputs. Output levels, in turn, are viewed as determined by the general equilibrium of the economy through the interaction of all markets. In this context, the (conditional) profit function is defined at given level of $\mathbf{y}$ on the production frontier

(2-10) $$\Pi^{Pt}(\mathbf{p},\mathbf{w};\mathbf{y}) = \mathbf{p}\cdot\mathbf{y} - \mathrm{C}^{t}(\mathbf{w},\mathbf{y}),$$

where we $\Pi^{Pt}(\mathbf{p},\mathbf{w};\mathbf{y})$ is the partial-equilibrium “Paretian” net-profit function, which is conditional on the level of the outputs $\mathbf{y}$.

The “Paretian” net-profit function (2-10) may have a value between those of the short-run “Paretian” net profit function” (where the outputs are fixed at a given level $\overline{\mathbf{y}}$) and the long-run equilibrium profit function that is

(2-11) $$\Pi^{Pt}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}}) \le \Pi^{Pt}(\mathbf{p},\mathbf{w};\mathbf{y}) \le \Pi^{t}(\mathbf{p},\mathbf{w}).$$

The difference between $\Pi^{M^t}(\mathbf{p},\mathbf{w};\bar{\mathbf{y}})$ and $\Pi^{M^t}(\mathbf{p},\mathbf{w};\mathbf{y})$ lay in the fact that, although $\bar{\mathbf{y}}$ and $\mathbf{y}$ are both assumed to be on the production frontier, the former is fixed and remains unchanged, whereas the latter is quasi-fixed and is allowed to change towards its optimal level.

The two perspectives of "Marshallian" and "Paretian" net profit functions that we have just defined do not appear to belong to the general concept of McFadden's (1978) "restricted profit function" represented by (2-7), but are, instead, more general. Despite the name given by McFadden to his formula of "restricted profit function", this includes only the unrestricted profit function as special case, whereas it includes the conditional cost and revenue functions as other special cases. By contrast, the Marhallian and Paretian net profit function consider cases where the producer has, respectively, the inputs and outputs fixed or exogenously determined. These two functions are defined by following Lau's (1972, p. 281, fn. 3) suggestion, referring to profits *net* of total costs. Therefore, they are different from the standard concept of "short-run (or restricted) (gross) profit function" which corresponds to the concept of a revenue function from which the costs of fixed or given inputs are not subtracted.

2.3 The envelope theorem

The relationship between the general conditional net-profit function with profit, revenue, and cost functions in the short-run disequilibrium is described by the following theorem:

THEOREM 2.1. Envelope theorem. *The long-run equilibrium profit function* $\Pi^t(\mathbf{q},\mathbf{p}_k)$ *is the higher envelope of the general conditional net-profit function* $\Pi^{McF^t}(\mathbf{q},\bar{\mathbf{k}})$ *in the space of output and input prices, with some points in common, that is*

(2-12) $$\Pi^{McF^t}(\mathbf{q},\bar{\mathbf{k}}) \le \Pi^t(\mathbf{q},\mathbf{p}_k)$$

where $\bar{\mathbf{k}} = \mathbf{k}^t(\bar{\mathbf{q}},\bar{\mathbf{p}}_k)$, *and the prices* $(\bar{\mathbf{q}},\bar{\mathbf{p}}_k)$ *are those that make compatible the levels of* $\bar{\mathbf{k}}$ *with the long-run equilibrium demand or supply functions.*

In terms of the short-run equilibrium Marshallian and Paretian net-profit functions using the respective translations given by (*I*), (*II*), or (*III*), the results of Theorem (2.1) are the following:

(2-13) $$\Pi^{M^t}(\bar{\mathbf{p}},\bar{\mathbf{w}};\bar{\mathbf{x}})) = \Pi^t(\bar{\mathbf{p}},\bar{\mathbf{w}})$$

$$\mathrm{R}^t(\bar{\mathbf{p}},\bar{\mathbf{x}}) - \bar{\mathbf{w}}\cdot\bar{\mathbf{x}} = \mathrm{R}^t(\bar{\mathbf{p}},\bar{\mathbf{x}}) - \mathrm{C}^t(\bar{\mathbf{w}},\bar{\mathbf{y}})$$

using (2-6), (2-8), and Hotelling's lemma

(2-14) $$\Pi^{P^t}(\bar{\mathbf{p}},\bar{\mathbf{w}};\bar{\mathbf{y}})) = \Pi^t(\bar{\mathbf{p}},\bar{\mathbf{w}})$$

$$\overline{\mathbf{p}}\cdot\overline{\mathbf{y}} - \mathrm{C}^t(\overline{\mathbf{w}},\overline{\mathbf{y}}) = \mathrm{R}^t(\overline{\mathbf{p}},\overline{\mathbf{x}}) - \mathrm{C}^t(\overline{\mathbf{w}},\overline{\mathbf{y}})$$

using (2-6), (2-10), and Hotelling's where $\overline{\mathbf{x}} = \mathbf{x}(\overline{\mathbf{p}},\overline{\mathbf{w}})$ and $\overline{\mathbf{y}} = \mathbf{y}(\overline{\mathbf{p}},\overline{\mathbf{w}})$, and

(2-15) $$\Pi^{Mt}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}}) \le \Pi^t(\mathbf{p},\mathbf{w})$$

$$R^t(\mathbf{p},\overline{\mathbf{x}}) - \mathbf{w}\cdot\overline{\mathbf{x}} \le R^t(\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})) - C^t(\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w}))$$

(2-16) $$\Pi^{Pt}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}}) \le \Pi^t(\mathbf{p},\mathbf{w}) \text{ and}$$

$$\mathbf{p}\cdot\overline{\mathbf{y}} - C^t(\mathbf{w},\overline{\mathbf{y}}) \le R^t(\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})) - C^t(\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w}))$$

The relations (2-15) and (2-16) reveal that $\Pi^t(\mathbf{p},\mathbf{w})$ is the higher envelope of $\Pi^{Mt}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}})$ and $\Pi^{Pt}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}})$, respectively, in the space of output and input prices. The inequalities (2-15) and (2-16) imply that $\Pi^t(\mathbf{p},\mathbf{w})$ is an upper bound. Equation (2-13) and (2-14) reveal that the long-run equilibrium profit function and the conditional Marshallian and Paretian profit functions share some points in common. These common points are in correspondence with those input and output bundles that maximize profits in the long-run equilibrium. The results given above are related to those obtained by Diewert's (1974, p. 140, Lemma I).

The envelope relationship between the Marshallian and Paretian net profit functions, on one part, and the full long-run equilibrium profit function, on the other, holds since the expressions (2-13) and (2-14) imply that each of the two first net profit functions never intersect the long-run equilibrium profit function.

3. A Net Profit Based Measure of Technical Change

We have noted earlier that the transformation function $\Pi^t(\mathbf{p},\mathbf{w})$ defined in the price space is homogeneous of degree one in its arguments, that is $\lambda\Pi^t(\mathbf{p},\mathbf{w}) = \Pi^t(\lambda\mathbf{p},\lambda\mathbf{w})$, and is dual to the transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ defined in the quantity space of outputs and inputs. Lau (1972, pp. 282-283) has clarified that this last function is almost homogeneous (see Aczel, 1966 for a discussion of almost homogeneous functions) in the sense that, by extending Euler's theorem, it is possible to obtain

(3-1) $$T^t(\mathbf{y},\mathbf{x}) = \lambda^0 T^t(\mathbf{y},\mathbf{x}) = T^t(\lambda^k\mathbf{y},\lambda\mathbf{x})$$

for every $\lambda \ne 0$. Thus, note that T^t is homogeneous of degree zero in $(\mathbf{y},\mathbf{x})$ and, if $k \ne 1$, then there are nonconstant returns to scale. Lau (1972, p. 284) showed that, in the equilibrium, $k = C(\mathbf{w},\mathbf{y})/R(\mathbf{p},\mathbf{x})$. Therefore, the equality $\Pi^t(\mathbf{p},\mathbf{w}) = 0$ holds identically when $k = 1$.

The transformation function $T^t(\mathbf{y},\mathbf{x})$ can be re-expressed in a form where the input quantities have negative values and the change in outputs is split into two parts: the first part is the change in outputs indexed by λ due to the change in inputs by the same proportion and the second part is the change in outputs indexed by $\lambda(\lambda^{k-1}-1)$ due to the scale effects, so that we have, for every $\lambda \ne 0$:

(3-2) $$T^{*t}(\mathbf{s},\mathbf{y},-\mathbf{x}) \;=\; \lambda^{-1}\,T^{*t}(\lambda\mathbf{s},\lambda\mathbf{y},\lambda(-\mathbf{x})),$$

where $\mathbf{s}=(\lambda^{k-1}-1)\mathbf{y}$ is a vector of implicit variables that satisfy the equality $T^t(\lambda^k\mathbf{y},\lambda\mathbf{x})=T^t(\lambda(\mathbf{s}+\mathbf{y}),\lambda\mathbf{x})$ and represent the internal or external scale factor affecting the levels of outputs, with $\lim_{\lambda\to 0}(\mathbf{s}+\mathbf{y})=\mathbf{y}$ and $\mathbf{s}=\mathbf{0}_M$ if $k=1$. The function T^{*t} is homogeneous of degree one in $(\mathbf{s},\mathbf{y},-\mathbf{x})$. If it is also separable from technical change, and this is input-output neutral in the sense that $T^t\equiv\sigma^t T=\sigma^t T^*$, then T^* is an aggregator function of the quantities of outputs, (negative) inputs, and scale effects. By transitivity, also T is an aggregator function of $(\mathbf{y},\mathbf{x})$ and scale effects. This result reveals that, under separability conditions, an almost homogeneous function is an aggregator of their explicit *and* implicit arguments. Based on this conclusion, we can establish the following theorem:

THEOREM 3-3 (*Netput* quantity aggregation): A well-defined production possibility frontier (3.1) characterized by an input-output neutral technical change can be considered as a *netput* quantity aggregator function of all output quantities and (negative) input quantities, net of possible scale induced input-output substitution effects.

Using duality theory, the following result is immediate:

THEOREM 3-4 (*Netput* price aggregation): A net profit function that is dual to a production possibility frontier (3.1) characterized by input-output neutral technical change can always be considered as a *netput* price aggregator function of all output and input prices taking into account scale induced input-output substitution effects.

Because of the duality between $\Pi^t(\mathbf{p},\mathbf{w})$ and $T^t(\mathbf{y},\mathbf{x})=0$, and $T^*[\lambda S,\lambda\mathbf{y},\lambda(-\mathbf{x})]=0$, these two theorems point us in the direction of an avenue that has been seldom travelled,[11] probably because the null value of the two price and quantity transformation functions does not allow the construction of index numbers in terms of ratios in contrast, for example, with the conditional revenue and cost functions. In the previous section we have defined the Marshallian net profit function (2-8), which is conditional on a fixed or given levels of inputs, and the Paretian net profit function (2-10), which is conditional on a fixed or given levels of outputs. These two net profit functions correspond to those suggested by Lau. We recall that they are linearly homogeneous not only in prices, but also in the fixed or given quantity levels of inputs and outputs, respectively.

For the sake of generality, let us consider the Paretian net profit function normalized with the value of the given output, say y_1^t, as (in order to save notation, we omit the explicit reference to the given output)[12]

[11] The exceptions include Archibald (1977), Balk (1998).

[12] The terminology "normalized profit" is originally due to Jorgenson and Lau (1974a)(1974b) (see also Lau, 1978). See Luenberger (1995, pp. 77-78) for more on the normalized profit function.

(3-5) $$\widetilde{\Pi}^t(\widetilde{\mathbf{p}},\widetilde{\mathbf{w}}) \equiv \Pi^t(\mathbf{p},\mathbf{w})/p_1 \cdot y_1$$
$$\equiv \Pi^t(\widetilde{\mathbf{p}},\widetilde{\mathbf{w}})/y_1$$

with $[1\ \widetilde{\mathbf{p}}] \equiv \mathbf{p}/p_1$, where $\widetilde{\mathbf{p}} \equiv [\frac{p_2}{p_1}\ \frac{p_3}{p_1} ... \frac{p_M}{p_1}]$, and $\widetilde{\mathbf{w}} \equiv \mathbf{w}/p_1$, where $\widetilde{\mathbf{w}} \equiv [\frac{w_1}{p_1}\ \frac{w_2}{p_1} ... \frac{w_N}{p_1}]$. Applying Hotelling's lemma to $\Pi^t(\mathbf{p},\mathbf{w})$ and normalizing by y_1, we have

(3-6) $$[1\ \widetilde{\mathbf{y}}] \equiv \mathbf{y}/y_1 = \nabla_{\mathbf{p}}\Pi^t(\mathbf{p},\mathbf{w})/y_1,$$
$$-\widetilde{\mathbf{x}} \equiv -\mathbf{x}/y_1 = \nabla_{\mathbf{w}}\Pi^t(\mathbf{p},\mathbf{w})/y_1$$

where $\widetilde{\mathbf{y}} \equiv [\frac{y_2}{y_1}\ \frac{y_3}{y_1} ... \frac{y_M}{y_1}]'$, and $\widetilde{\mathbf{x}} \equiv [\frac{x_1}{y_1}\ \frac{x_2}{y_1} ... \frac{x_N}{y_1}]'$; and therefore, $\widetilde{y}_i = \frac{\partial\widetilde{\Pi}^t}{\partial\widetilde{p}_i} = \frac{d\widetilde{\Pi}^t}{d\Pi^t}\cdot\frac{\partial\Pi^t}{\partial p_i}\cdot\frac{dp_i}{d\widetilde{p}_i}$ and $-\widetilde{x}_i = \frac{\partial\widetilde{\Pi}^t}{\partial\widetilde{w}_i} = \frac{d\widetilde{\Pi}^t}{d\Pi^t}\cdot\frac{\partial\Pi^t}{\partial w_i}\cdot\frac{dw_i}{d\widetilde{w}_i}$. We note that $[\widetilde{\mathbf{y}}\ \widetilde{\mathbf{x}}]'$ is the vector of the technical input-output coefficients referred to the production of one unit of y_1. Moreover, Π^t is linearly homogeneous in its price arguments by construction, *i.e.* $\lambda\Pi^t(\mathbf{p},\mathbf{w}) = \Pi^t(\lambda\mathbf{p},\lambda\mathbf{w})$, so that $\Pi^t(\mathbf{p},\mathbf{w}) = \mathbf{p}\nabla_{\mathbf{p}}\Pi + \mathbf{w}\nabla_{\mathbf{w}}\Pi$ by Euler's theorem. By contrast, $\widetilde{\Pi}$ is not linearly homogeneous in $(\widetilde{\mathbf{p}},\widetilde{\mathbf{w}})$, since $\widetilde{\Pi}^t(\widetilde{\mathbf{p}},\widetilde{\mathbf{w}}) = 1 + \widetilde{\mathbf{p}}\nabla_{\widetilde{\mathbf{p}}}\widetilde{\Pi} + \widetilde{\mathbf{w}}\nabla_{\widetilde{\mathbf{w}}}\widetilde{\Pi}$. These properties will be useful for the following discussion..

We decompose the observed absolute difference $(\widetilde{\Pi}^1 - \widetilde{\Pi}^0)$ into price and technical change components as follows:

(3-7) $$\widetilde{\Pi}^1(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1) - \widetilde{\Pi}^0(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0) = P_\pi^{0,1} + T_\pi^{0,1}$$

where $P_\pi^{0,1} \equiv$ price change component, and $T_\pi^{0,1} \equiv$ technical change component. In the general non-homothetic case, these two components cannot be univocally determined and alternative measures can be defined as follows

(3-8) $$P_{P-\pi}^{0,1} \equiv \widetilde{\Pi}^1(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1) - \widetilde{\Pi}^1(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0)$$ (Paasche-weighted price component)

(3-9) $$T_{L-\pi}^{0,1} \equiv \widetilde{\Pi}^1(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0) - \widetilde{\Pi}^0(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0)$$ (Laspeyres-weighted technical change component)

or,

(3-10) $$P_{L-\pi}^{0,1} \equiv \widetilde{\Pi}^0(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1) - \widetilde{\Pi}^0(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0)$$ (Laspeyres-weighted price component)

(3-11) $$T_{P-\pi}^{0,1} \equiv \widetilde{\Pi}^1(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1) - \widetilde{\Pi}^0(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1)$$ (Paasche-weighted technical change component)

The indicators $T_{L-\pi}^{0,1}$ and $T_{P-\pi}^{0,1}$ are alternative measures of the relative rate of technical change between the observation points $t=0$ and $t=1$.

In addition to the components of the measures defined in (3-7) through (3-11) that can be evaluated using observed data for periods 0 and 1, some way must be found to evaluate the two hypothetical terms: $\widetilde{\Pi}^1(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0)$ and $\widetilde{\Pi}^0(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1)$. The first of these hypothetical terms represents the net profit that would have resulted from carrying out the production using the period 1 technology with the period 0 prices, and the second of these represents the net profit that would have resulted from carrying out the production using the period 0 technology with the period 1 prices.

One approach to evaluating these hypothetical quantities is to assume that the technical input-output coefficients are fixed – i.e., that they are the same – in periods 0 and 1. We can then use observable data to evaluate the hypothetical values as:

(3-12 $$\widetilde{\Pi}^0(\widetilde{\mathbf{p}}^1,\widetilde{\mathbf{w}}^1) = 1+\widetilde{\mathbf{p}}^1\widetilde{\mathbf{y}}^0 - \widetilde{\mathbf{w}}^1\widetilde{\mathbf{x}}^0 \quad \text{with the technology at } t=0,$$

or

(3-13) $$\widetilde{\Pi}^1(\widetilde{\mathbf{p}}^0,\widetilde{\mathbf{w}}^0) = 1+\widetilde{\mathbf{p}}^0\widetilde{\mathbf{y}}^1 - \widetilde{\mathbf{w}}^0\widetilde{\mathbf{x}}^1 \quad \text{with the technology at } t=1,$$

If we cannot assume that the input-output coefficients are fixed, then the following sort of approach might be adopted for evaluating the hypothetical quantities.

Let us consider the quadratic mean-of-order-*r* functional form established by Diewert (1976):

(3-14) $$\Pi^t_{Q_r}(\mathbf{z}) \equiv \left[\sum_{i=1}^{K}\sum_{j=1}^{K}\alpha^t_{ij}(z_i z_j)^{r/2}\right]^{\frac{1}{r}} \qquad \text{for } K=(M+N) \text{ and } r\neq 0,$$

where $\mathbf{z}^t \equiv [\mathbf{p}^t\ \mathbf{w}^t]$, $\alpha^t_{ij}=\alpha^t_{ji}$ and α^t_{ij} are allowed to change over t reflecting technical progress. If technical change is jointly Hicks-profit neutral, the function (3-14) becomes

(3-15) $$\Pi^t_{Q_r}(\mathbf{z}) \equiv \sigma^t\left[\sum_{i=1}^{K}\sum_{j=1}^{K}\alpha_{ij}(z_i z_j)^{r/2}\right]^{\frac{1}{r}}$$

where σ^t is a technological parameter[13]. Using the notation $\widetilde{\mathbf{z}} \equiv [1\ \widetilde{\mathbf{p}}\ \widetilde{\mathbf{w}}]$ with $\widetilde{\mathbf{p}} \equiv [\frac{p_2}{p_1}\frac{p_3}{p_1}...\frac{p_M}{p_1}]$ and $\widetilde{\mathbf{w}} \equiv [\frac{w_1}{p_1}\frac{w_3}{p_1}...\frac{w_N}{p_1}]$, the normalized profit function defined as $\widetilde{\Pi}^t_{Q_r}(\widetilde{\mathbf{z}}) \equiv \widetilde{\Pi}^t_{Q_r}(\mathbf{z})/p_1 = \Pi^t_{Q_r}(\mathbf{z})/p_1 y_1$ can be derived from (3-14) as

(3-16) $$\widetilde{\Pi}^t_{Q_r}(\widetilde{\mathbf{z}}) \equiv \left[\alpha^t_{11} + 2\sum_{j=2}^{K}\alpha^t_{1j}\widetilde{z}^{r/2}{}_j\sum_{i=2}^{K}\sum_{j=2}^{K}\alpha^t_{ij}(\widetilde{z}_i\widetilde{z}_j)^{r/2}\right]^{\frac{1}{r}}\cdot\frac{1}{y^t_1}$$

or from (3-15), in the case of Hicks-profit neutral technical progress, as

(3-17) $$\widetilde{\Pi}^t_{Q_r}(\widetilde{\mathbf{z}}) \equiv \sigma^t\left[\alpha_{11} + 2\sum_{j=2}^{K}\alpha_{1j}\widetilde{z}^{r/2}{}_j\sum_{i=2}^{K}\sum_{j=2}^{K}\alpha_{ij}(\widetilde{z}_i\widetilde{z}_j)^{r/2}\right]^{\frac{1}{r}}\cdot\frac{1}{y^t_1}$$

[13] See Chandler (1988, pp. 224-228) for different types of technical change with profit functions.

If the two functions $\widetilde{\Pi}^t_{Q_r}(\widetilde{\mathbf{z}})$ with t = 0,1 have the functional form (3-16) or (3-17) with $r=1,$, corresponding to the *quadratic mean-of-order-1* (Generalized Leontief) profit function, then (see Diewert, 2005 and Milana, 2005)

$$\text{(3-18)} \qquad P^{0,1}_{Q_1} \equiv \sum_{i=2}^{K} \left\{ \frac{(\widetilde{z}_i^0)^{\frac{1}{2}} \widetilde{q}_i^0}{(\widetilde{z}_i^0)^{\frac{1}{2}} + (\widetilde{z}_i^1)^{\frac{1}{2}}} + \frac{(\widetilde{z}_i^1)^{\frac{1}{2}} \widetilde{q}_i^1}{(\widetilde{z}_i^0)^{\frac{1}{2}} + (\widetilde{z}_i^1)^{\frac{1}{2}}} \right\} (\widetilde{z}_i^1 - \widetilde{z}_i^0)$$

where $\widetilde{\mathbf{q}}^t \equiv [1 \ \ \widetilde{\mathbf{y}}^t \ (-\widetilde{\mathbf{x}}^t)]'$. with $\widetilde{\mathbf{y}}^t \equiv [\frac{y_2^t}{y_1^t} \frac{y_3^t}{y_1^t} ... \frac{y_M^t}{y_1^t}]$ and $\widetilde{\mathbf{x}}^t \equiv [\frac{x_1^t}{y_1^t} \frac{x_2^t}{y_1^t} ... \frac{x_N^t}{y_1^t}]$.

If the two profit functions $\widetilde{\Pi}^t_{Q_r}(\mathbf{z})$ with t = 0,1 have the functional form (3-16) or (3-17) with $r=2,$ corresponding to the *quadratic mean-of-order-2* (Konüs-Byushgens) profit function, then (see Diewert, 2005 and Milana, 2005)

$$\text{(3-19)} \qquad P^{0,1}_{Q_2} \equiv \sum_{i=2}^{K} \left\{ \frac{\widetilde{\Pi}^0_{Q_2} \cdot \widetilde{q}_i^0}{\widetilde{\Pi}^0_{Q_2} + \widetilde{\Pi}^1_{Q_2}} + \frac{\widetilde{\Pi}^1_{Q_2} \cdot \widetilde{q}_i^1}{\widetilde{\Pi}^0_{Q_2} + \widetilde{\Pi}^1_{Q_2}} \right\} (\widetilde{z}_i^1 - \widetilde{z}_i^0)$$

and $\widetilde{\Pi}^t_{Q_2} \equiv \mathbf{z}^t \cdot \mathbf{q}^t$ with $t = 0,1$.

It is possible to show that, with any pair of quadratic mean-of-order-*r* functional forms differing in parameters as in (3-16) or (3-17), the indicator (3-19) is a weighted average of the two ratios of Laspeyres and Paasche-weighted profit values evaluated only with changes in prices

$$\text{(3-20)} \qquad P^{0,1}_{Q_r} = (1-\lambda) P^{0,1}_{L-Q_r} + \lambda \, P^{0,1}_{P-Q_r}$$

where $P^{0,1}_{L-Q_r} \equiv \widetilde{\Pi}^0_{Q_r}(\widetilde{\mathbf{z}}^1) - \widetilde{\Pi}^0_{Q_r}(\widetilde{\mathbf{z}}^0)$

$P^{0,1}_{P-Q_r} \equiv \widetilde{\Pi}^1_{Q_r}(\widetilde{\mathbf{z}}^1) - \widetilde{\Pi}^1_{Q_r}(\widetilde{\mathbf{z}}^0)$

and

$$\text{(3-21)} \qquad \lambda \equiv \frac{\frac{1}{2}(\widetilde{\Pi}^1_{Q_r} - \widetilde{\Pi}^0_{Q_r}) + \frac{1}{4}\sum_{i=2}^{K}\sum_{j=2}^{K}(\widetilde{z}_i^1 - \widetilde{z}_i^1)(\alpha^1_{ij} - \alpha^0_{ij})(\widetilde{z}_j^1 - \widetilde{z}_j^1)}{(\widetilde{\Pi}^1_{Q_r} - \widetilde{\Pi}^0_{Q_r})},$$

where $A^r \equiv [\alpha^t_{ij}]$ (with $i,j = 2,...,K$) is the symmetric matrix of second-order parameters of the function. If these parameters are constant over the examined period, that is $A^1 = A^0$, which is the case of profit-neutral technical progress, then $\lambda = 1/2$.

Using (3-7), the residual component representing the relative rate of technical change is given by

$$(3\text{-}22) \qquad T_{Q_r}^{0,1} \equiv [\widetilde{\Pi}_{Q_r}^{1}(\widetilde{\mathbf{z}}^1) - \widetilde{\Pi}_{Q_r}^{0}(\widetilde{\mathbf{z}}^0)] - P_{Q_r}^{0,1}$$

In applying the index number formulas presented here that allow us to overcome the input-output non-homothetic separability, we shall assume that technical change is Hicks-neutral, reflecting homothetic changes in parameters. Thus, if technical change is not Hicks-neutral (a number of empirical studies seem to confirm this hypothesis[14]), then our measure of technical change does not provide us with an "aggregate" of technical effects. In this case, our indicator of price changes is not homogeneous of degree zero with respect to parameter changes, reflecting its dependence on the particular technology path between the two situations under comparison while the technology change indicator is not homogeneous of degree one thus reflecting a distorsion.

Finally, some explanation must be given in order to relate the relative rate of technical change obtained with (3-22) to the traditional indicator of total factor productivity growth (*TFPG*). The last one is usually represented with the rate of change in the ratio between aggregated outputs to aggregated inputs. In the stylized model of one output (y) and one input (x), the relative rate of *TFPG* between $t=0$ and $t=1$ with respect to the situation $t=0$ is given by

$$(3\text{-}23) \qquad TFPG^0 \equiv \left[\frac{y^1}{x^1} - \frac{y^0}{x^0}\right] / \frac{y^0}{x^0} = \frac{y^1x^0 - y^0x^1}{y^0x^1}$$

Similarly, we could define the relative rate of change in *TFP* with respect to the comparison situation $t=1$:

$$(3.24) \qquad TFPG^1 \equiv -\left[\frac{y^0}{x^0} - \frac{y^1}{x^1}\right] / \frac{y^1}{x^1} = \frac{y^1x^0 - y^0x^1}{y^1x^0}$$

In the same stylized model of one output and one input in a profit-maximizing production unit operating with constant returns to scale in a competitive market, from (3-7), we have

$$(3\text{-}25) \qquad T_{P-\pi}^{0,1} = [\widetilde{\Pi}^1(1,\widetilde{w}^1) - \widetilde{\Pi}^0(1,\widetilde{w}^0)] - [\widetilde{\Pi}^0(1,\widetilde{w}^1) - \widetilde{\Pi}^0(1,\widetilde{w}^0)]$$

$$= [\widetilde{\Pi}^1(1,\widetilde{w}^1) - \widetilde{\Pi}_r^0(1,\widetilde{w}^1)] \quad \text{(which is Paasche-type price-weighted technical change component)}$$

where we recall that $\widetilde{\Pi}^t(1,\widetilde{w}^1) = 1 - \frac{x^t}{y^t}\widetilde{w}^1$ with $t = 0,1$. Therefore, we have

[14] See, for example, Takayama (1974), who found empirical evidence of a biased technical change in the U.S. in a paper that appeared on the same issue of the *AER* where Samuelson and Swamy's (1974) article was published. These authors had, instead, favoured the homotheticity hypothesis in production theory rather than in consumer theory. They claimed in fact: "Fortunately, in the case of production theory […] homotheticity is not always so unrealistic" (p. 577, fn. 10). Other examples of empirical evidence of a biased technical change are those of Jorgenson and Fraumeni (1981) and Jorgenson, Gollop, and Fraumeni (1987, 211-260) for the US.

(3-26) $$P_{L-\pi}^{0,1} \equiv [\widetilde{\Pi}^0(1,\widetilde{w}^1) - \widetilde{\Pi}^0(1,\widetilde{w}^0)] = \frac{x^0}{y^0} \cdot \left[\frac{w^0}{p^0} - \frac{w^1}{p^1} \right]$$

(3-27) $$T_{P-\pi}^{0,1} \equiv [\widetilde{\pi}^1(1,\widetilde{w}^1) - \widetilde{\pi}_r^0(1,\widetilde{w}^1)]$$

$$= (-\frac{x^1}{y^1} + \frac{x^0}{y^0})\frac{w^1}{p^1} = \frac{y^1x^0 - y^0x^1}{y^0y^1} \cdot \frac{w^1}{p^1}$$

$$= \frac{y^1x^0 - y^0x^1}{y^0x^1} \cdot \frac{w^1}{p^1} / \frac{y^1}{x^1}$$

$$= \frac{y^1x^0 - y^0x^1}{y^0x^1}, \quad \text{since } \frac{w^1}{p^1} = \frac{y^1}{x^1},$$

$$= TFPG^0 \quad \text{(as defined by (3-23))}$$

Similarly, by defining a Laspeyres-type price-weighted technical change component, we obtain $T_{L-\pi}^{0,1} = (-\frac{x^1}{y^1} + \frac{x^0}{y^0})\frac{w^0}{p^0} = \frac{y^1x^0 - y^0x^1}{y^1x^0} = TFPG^1$. Therefore, the more general model based on the normalized net profit function leads us directly to a measure of the *TFPG* relative rate.

If the returns to scale are non-constant and the profits are non-zero, we may assume a non-linear functional form for Π^t, as for example, a quadratic mean of order-2, from which we derive

(3-28) $$\widetilde{\Pi}_{Q_2}^t \equiv \left[\alpha_{11} + 2\alpha_{12}\left(\frac{w}{p}\right) + \alpha_{22}\left(\frac{w}{p}\right)^2 \right]^{\frac{1}{2}} \cdot \frac{1}{y}$$

from which we can derive

(3-29) $$P_{Q_2}^{0,1} \equiv \left[\frac{\widetilde{\Pi}_{Q_r}^0}{\widetilde{\Pi}_{Q_r}^0 + \widetilde{\Pi}_{Q_r}^0} \cdot \frac{x^0}{y^0} + \frac{\widetilde{\Pi}_{Q_r}^1}{\widetilde{\Pi}_{Q_r}^1 + \widetilde{\Pi}_{Q_r}^1} \cdot \frac{x^1}{y^1} \right] \cdot \left[\frac{w^0}{p^0} - \frac{w^1}{p^1} \right]$$

(3-30) $$T_{Q_2}^{0,1} \equiv [\widetilde{\Pi}_{Q_r}^1 - \widetilde{\Pi}_{Q_r}^0] - P_{Q_2}^{0,1}$$

We note that (3-27) and (3-28) are weighted averages of Laspeyres-type and Paasche weighted price and technical change components

The last two formulas could be contrasted with the respective formulas (3-26) and (3-27), which are traditionally used with the hypothesis of input-output strong separability, implying constant returns to scale. This is in fact usually done in the case of multi-outputs and multi-inputs that are separately aggregated into one output and one input magnitudes. Our formulas (3-29) and (3-30) reveal that more flexible components can be defined in order to accommodate for price-induced input-output substitution due to non-constant returns to scale. This is what is done in the case of multi-outputs and multi-inputs aggregated separately under the hypothesis of weak separability, and allowing for an adjustment in the weights. Usually the adjustment parameter capturing the scale effects is found by means of parametric estimations. Here, we have established a completely non-parametric method. Moreover, if input-output weak separability is ruled out in the case of multi outputs and multi inputs, then the formulas (3-18) through (3-22) can be applied to disaggregated data. These are the decomposition formulas that have applied to the case of Italy described in the following section.

4. Empirical Results

> "*Previous studies have found diminishing internal returns but large productivity spillovers. Those findings appear to be largely an artifact of using value-added data rather than the correct gross-output data*"
>
> Susanto Basu and John G. Fernald (1992)
>
> "*A typical (roughly) two-digit industry in the United States appears to have constant or slightly decreasing returns to scale"*
>
> Susanto Basu and John G. Fernald (1996)

Italy has had some special reasons to be concerned about the productivity of the economy. Most raw materials including fuel must be imported, leaving the country vulnerable to external economic shocks especially in times unrest in the oil producing regions of the world. Over the past decade, Italy has pursued a tight fiscal policy in order to meet the requirements of the European Economic and Monetary Unions. Numerous short-term reforms aimed at improving competitiveness and long-term growth have been enacted. Nevertheless, the budget deficit has now exceeded the 3% European ceiling. Moreover, the economy experienced almost no growth in 2005, and unemployment is relatively high by European Union standards. The economy of Italy remains divided into a developed industrial north and a less-developed, welfare-dependent, agricultural south, where the unemployment rate has been chronically high and is currently in the 20% range.

Our proposed measures given in (3-14) through (3-18) can be applied using the Italian data from the *EU KLEMS* project.

The *EU KLEMS* database provides us with time series of price and quantity indexes of outputs and inputs within supply and use input-output tables at the level of disaggregation of 72 industries as well as services of different types of labour and durable capital goods used in production. It has been constructed in close collaboration with national statistical institutions and is fully consistent with the official national accounts, following the directives of Eurostat. Here, we present only the aggregate level results for Italy.

We have also evaluated several conventional cost-based *TFPG* indexes. Given the assumptions of input-output separability and constant returns to scale, the chosen conventional measures of *TFPG* are exact for the Leontief, Generalized Leontief, and Konüs-Byushgens cost functions .The results are shown in Table 1 in terms of relative rates of change. With the first functional form, the exact aggregating input-price index numbers are given by either direct Laspeyres or Paasche aggregating index numbers, whereas, with last two cost functions, the exact aggregating input-price index numbers are given by formula (3-11), where the parameter r is equal to 1 and 2, respectively. We recall that the input-price index number that is exact for the Generalized Leontief aggregating function is the implicit Walsh index number, whereas the index number that is exact for the Konüs-Byushgens aggregating function is the Fisher ideal index. The relative changes in productivity are, therefore, obtained implicitly from the changes in the inverse of total average nominal costs deflated by means of the one of the alternative input-price index numbers.

In Table 1, a wide variation in the Laspeyres-Paasche spread can be noted. A large spread may reveal that the true indicator (if it exists as an aggregate indicator) may be far from being close to the measures constructed here. The Paasche and Laspeyres indicators turn out to be very close during the years 1993-1997 and 2001-2003, but relatively far from each other during the decades of the seventies and the eighties (for similar results obtained for other countries, see also Fujikawa and Milana, 1996 and Milana, 2001). This is not surprising, considering the relatively intense restructuring activities that had taken place in Italy and other European countries after the first and second oil shocks. Intense technological change and price-induced input substitutions were reflected in the immediate reply of cost-reducing restructuring policies within the firms and governments in those periods.

Moreover, a reverse position in the ranking of numerical values of the Laspeyres- and Paasche-type indicators with respect to the indications of the theory of bounds of cost-based economic index numbers, may suggest that non-homotheticity has taken place in all years, except three. The implicit Laspeyres-type (implicit Paasche-type) quantity index, corresponding to the total nominal costs deflated by the direct Laspeyres (direct Paasche) input-price index, is, in fact, a direct Paasche-type (direct Laspeyres-type) input-quantity index . The theory of bounds that we have recalled above suggests that, in the homothetic case, the direct Laspeyres index (which is always the upper bound of the Laspeyres-weighted "true" index) is comparable to and higher than the direct Paasche index (which is always the lower bound of the Paasche-weighted "true" index). If, instead, the direct Laspeyres index turns out to be lower than the direct Paasche index, then a non-homothetic change situation may have occurred. In Table 1, the direct Laspeyres turns out to be substantially higher than the direct Paasche *TFP* index growth in only 2 years in the whole period 1971-2003, thus indicating that non-homotheticity effects have been the norm rather than the exception.

The Fisher ideal index number of *TFP*, which is exact corresponding to the Konüs-Byushgens (KB) aggregator function, is equal to the geometric average of the Laspeyres and Paasche-type indicators and, therefore, is always found between their bounds by construction. Moreover, the Fisher ideal and the direct Walsh index numbers of *TFP* are found to be very close to each other, thus confirming that they always perform in close approximation (see Hill, 2006b). Moreover, the fact that these two indicators are, respectively, a perfect and a close approximation to the geometric average of the two Laspeyers and Paasche indexes is problematic in the case of severe non-homothetic changes, where the "true" index is brought beyond the Laspeyres-Paasche interval in a very asymmetrical way.

The results obtained by considering a separable cost function based on the input-output separability assumption can be contrasted with those obtained with the indicators derived from a profit function in the input-output non-separability case. Figure 1, showing the technical-change measures obtained with the indicators based on the GL and KB cost functions, can be contrasted with Figure 2, showing the technical-change measures obtained with indicators based on the GL and KB profit functions. We must recognize that these results are not fully comparable, since the functional forms of the cost and profit functions are not "self-dual", meaning that the profit function corresponding to a GL (or KB) cost function does not have a GL (or KB) functional form, and vice versa. The consequence of this is that we are comparing the results obtained under different hypotheses on input-output separability *combined with* different hypotheses on functional forms. However, when the alternative cost-based indexes given in Table 1 do not differ widely, their difference with respect to the profit-based indexes may be mainly due to the different separability hypotheses.

Figure 3 compares the results obtained with cost- and profit-based indicators shown in Figures 1 and 2. We note that the cost-based *TFP* measure and the profit-based technical change measures are significantly different in many years of the examined period. In 11 out of 33 years, the difference has been found to be at least greater than 50 per cent. The productivity slowdown observed in Italy after the year 2000 seems to be reduced to more than half within the picture obtained with the more general framework in a period of reduced pace of economic growth. This should be contrasted with the years 1999 and 2000, where the higher dynamics of production has led the cost-based measure of *TFP* growth to be lower than the profit-based measure of technical change. These results suggest that the Italian economy is characterized by non-constant returns to scale and is affected by various institutional and environmental constraints that hinder the growth of the firms in size and full exploitation of national factor employment. The Italian productive economy is, in fact, based on small and medium firms, which, given the constraints to the growth in size, face decreasing returns to scale in many cases.

Figures 4 and 5 show, respectively, the effects that *TFP* and technical changes have brought about on real factor rewards during the period 2000-2003. It can be seen that, in the year 2000, the high increase in energy prices (notably crude oil prices) occurred during a worldwide economic expansion has absorbed the whole *TFP* gain achieved in that year and required losses in the real labour compensation and services. Moreover, the positive short-run performance in production has allowed some small gains in the real capital rewards (both ICT and non-ICT). These movements in real factor prices appear amplified in the more general framework based on profit-based indicators.

The same Figures 4 and 5 permit us also to contribute to the current debate on productivity slowdown in Italy. During the period 2001-2003, we observe that this productivity slowdown does not appear to be related to efficiency losses as much as they seem if we look at more traditional indicators. These seem to be theoretically unfounded since the hypotheses on input-output separability and constant returns to scale on which they are based are not, in fact, confirmed by the results obtained using more general models. Efficiency losses turn out to be negligible and the estimated productivity decline falls within the range of measurement errors.

8. Conclusion

A partial solution to the non-separability problem in technical change measurement may be found by aggregating outputs and inputs together using the so-called transformation functions. The profit function can be considered as a transformation function in the space of prices and may be used under the hypothesis that the observed data are optimal from the point of view of long-run equilibrium. A decomposition procedure has been devised to decompose changes in the value of the profit function into a technical change component and a price component without imposing any assumption on input-output separability.

The method proposed here has been applied empirically to the case of the Italian industries using the newly built database of the *EUKLEMS* project. Homotheticity seems to have been the exception rather than the rule in Italy during the period 1970-2003 and the results obtained have been contrasted with those of traditional approaches that assume input-output separability. Although these alternative measures are not fully comparable, we conclude that the *TFP* decline recently reported in Italy is not confirmed in size and direction by our findings on technical change.

Some important limitations still remain in this exercise, including the heroic assumption regarding the existence of conditional profit functions in the unworkable case of increasing returns to scale. The case of constant returns to scale is also problematic when dealing with undeterminable absolute quantity levels of all outputs and inputs, especially with producers that are price takers in input and output markets. A correction for short-run cyclical behaviour in production should also be made on the data concerning the services actually used of fixed or quasi-fixed factors.

Table 1. Alternative Measures of TFP Changes Based on Different Cost Functions (in percentage)

All industries in the Italian economy

Year	Implicit Laspeyres (direct Paasche) (1)	Implicit Konüs-Byushgens (ideal Fisher) (2)	Implicit Generalized Leontief (3)	Implicit Paasche (direct Laspeyres) (4)	Direct Paasche/Direct Laspeyres ratio (5) = (1)/(4)	Difference between direct Paasche and direct Laspeyres (6) = (1) - (4)
1971	0.65	0.47	0.48	0.30	2.20	0.35
1972	-1.33	-1.49	-1.8	-1.64	0.82	0.30
1973	2.93	2.86	2.86	2.78	1.05	0.15
1974	1.95	1.79	1.78	1.64	1.19	0.32
1975	-3.30	-3.45	-3.44	-3.61	0.91	0.31
1976	1.51	1.46	1.46	1.41	1.07	0.11
1977	-0.61	-0.65	-0.65	-0.68	0.89	0.07
1978	-0.06	-0.12	-0.12	-0.17	0.34	0.11
1979	-0.82	-0.93	-0.93	-1.05	0.78	0.23
1980	0.58	0.35	0.35	0.12	4.86	0.46
1981	-1.46	-1.50	-1.50	-1.54	0.94	0.09
1982	-0.70	-0.71	-0.71	-0.72	0.97	0.02
1983	0.17	0.14	0.14	0.12	1.35	0.04
1984	0.22	0.21	0.21	0.19	1.15	0.03
1985	1.68	1.66	1.66	1.63	1.03	0.05
1986	0.60	0.64	0.64	0.68	0.88	-0.08
1987	0.56	0.49	0.49	0.43	1.32	0.14
1988	1.00	0.98	0.98	0.95	1.05	0.05
1989	0.29	0.26	0.26	0.24	1.23	0.05
1990	-0.32	-0.35	-0.35	-0.38	0.83	0.06
1991	-0.34	-0.31	-0.31	-0.28	1.23	-0.06
1992	0.93	0.89	0.88	0.84	1.11	0.09
1993	0.94	0.94	0.94	0.94	1.00	0.00
1994	1.65	1.64	1.64	1.63	1.01	0.02
1995	1.20	1.20	1.20	1.21	0.99	-0.02
1996	-0.26	-0.26	-0.26	-0.26	1.00	0.00
1997	0.54	0.52	0.52	0.50	1.07	0.03
1998	-0.29	-0.30	-0.30	-0.30	0.97	0.01
1999	-0.08	-0.09	-0.09	-0.10	0.79	0.02
2000	0.73	0.63	0.62	0.53	1.36	0.19
2001	-0.31	-0.31	-0.31	-0.31	0.98	0.01
2002	-0.34	-0.34	-0.34	-0.35	0.96	0.01
2003	-0.42	-0.42	-0.42	-0.42	0.99	0.00

Figure 1. Technical-change measures based on GL and KB cost functions

All industries in the Italian economy

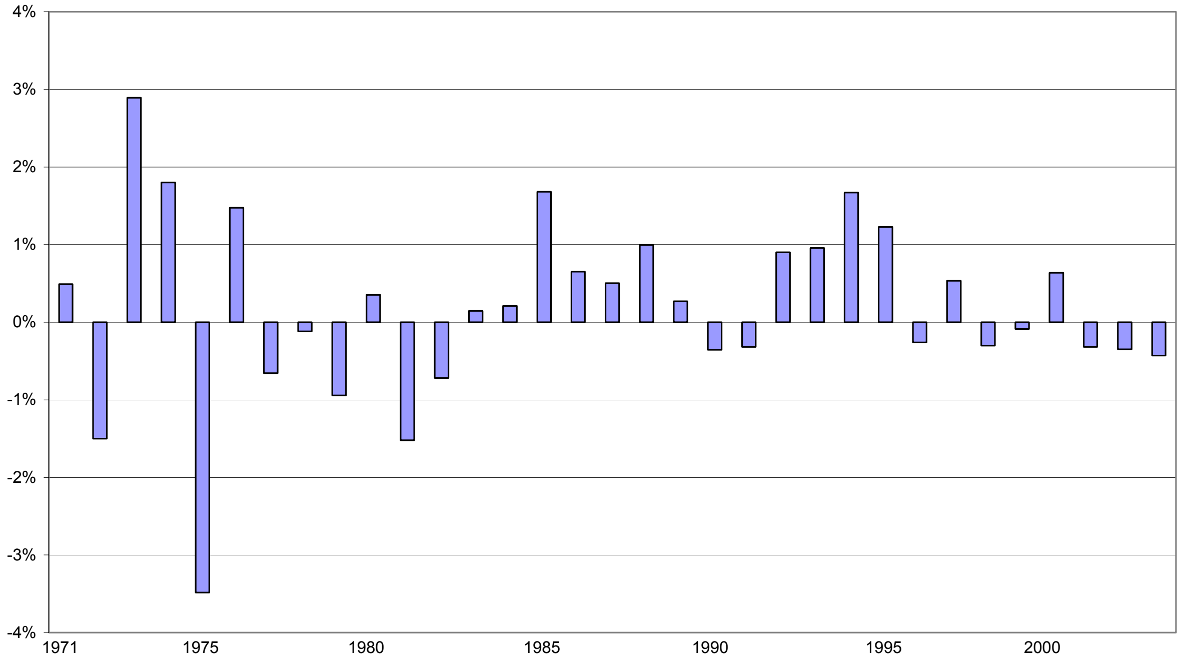


Figure 2. Technical-change measures based on GL and KB profit functions

All industries in the Italian economy

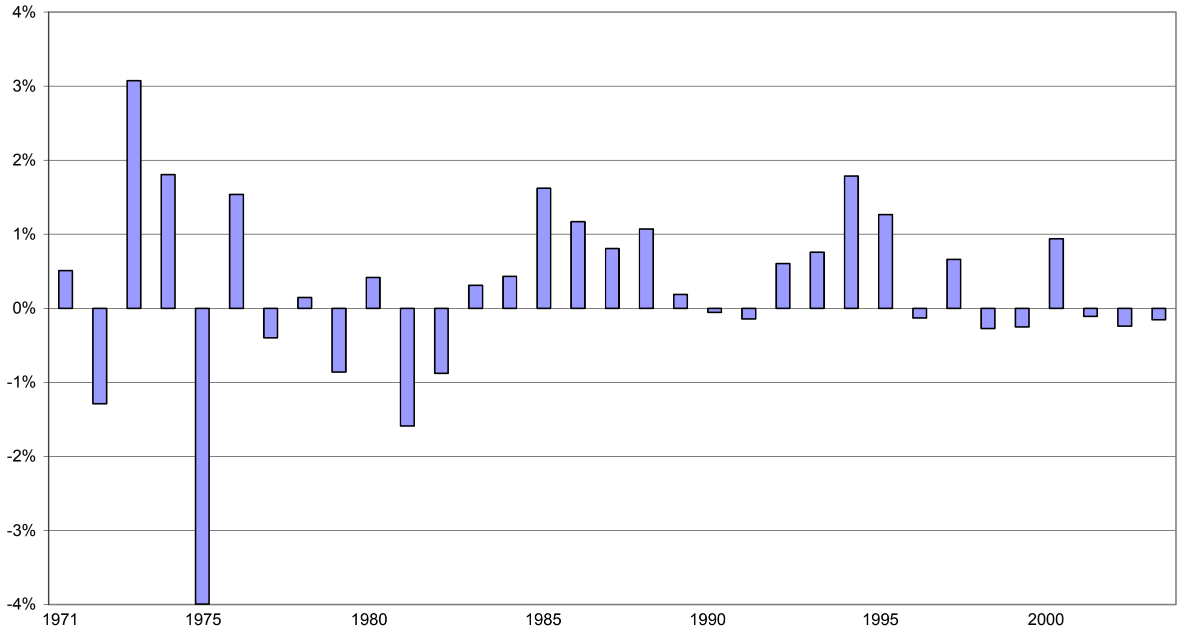


Figure 3. Relative differences bewteen GL (or KB) cost- and profit-based measures of TFP

All industries in the Italian economy

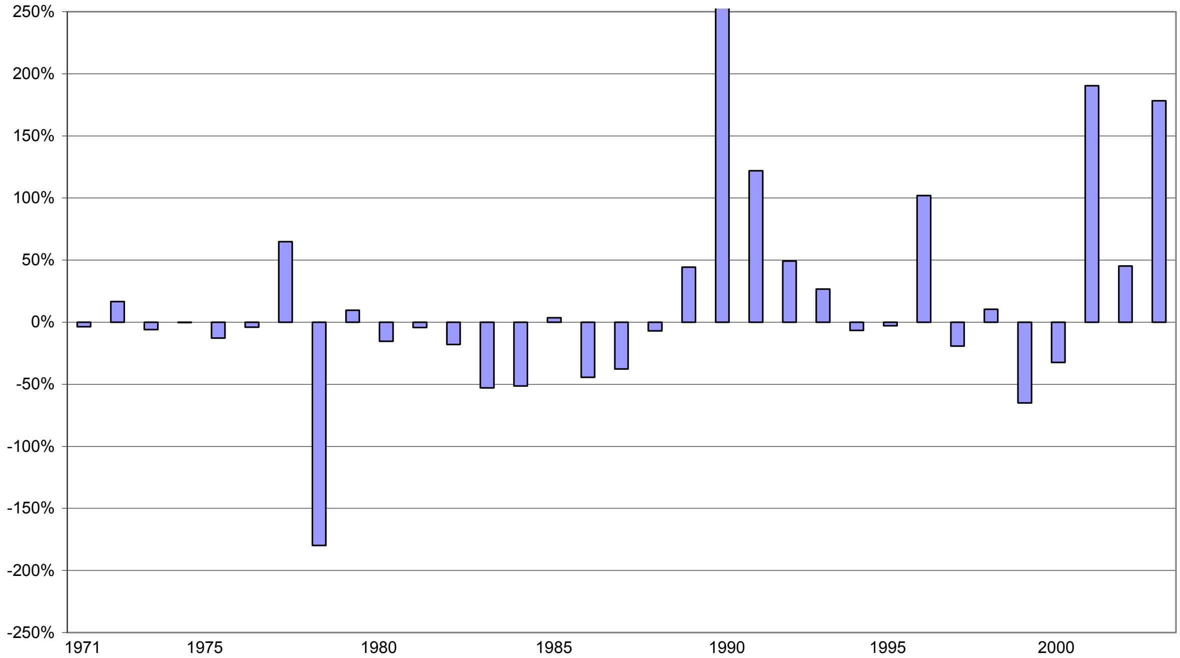


Figure 4. Measures of effects of TFP growth on real factor prices, based on the GL and KB cost functions

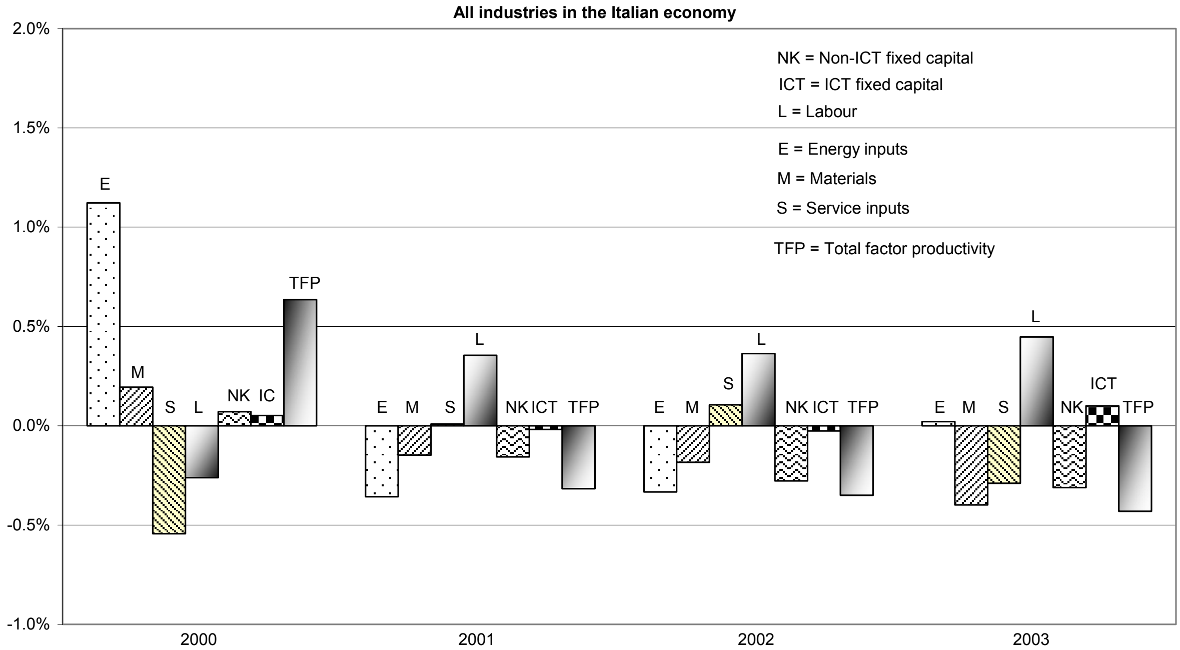
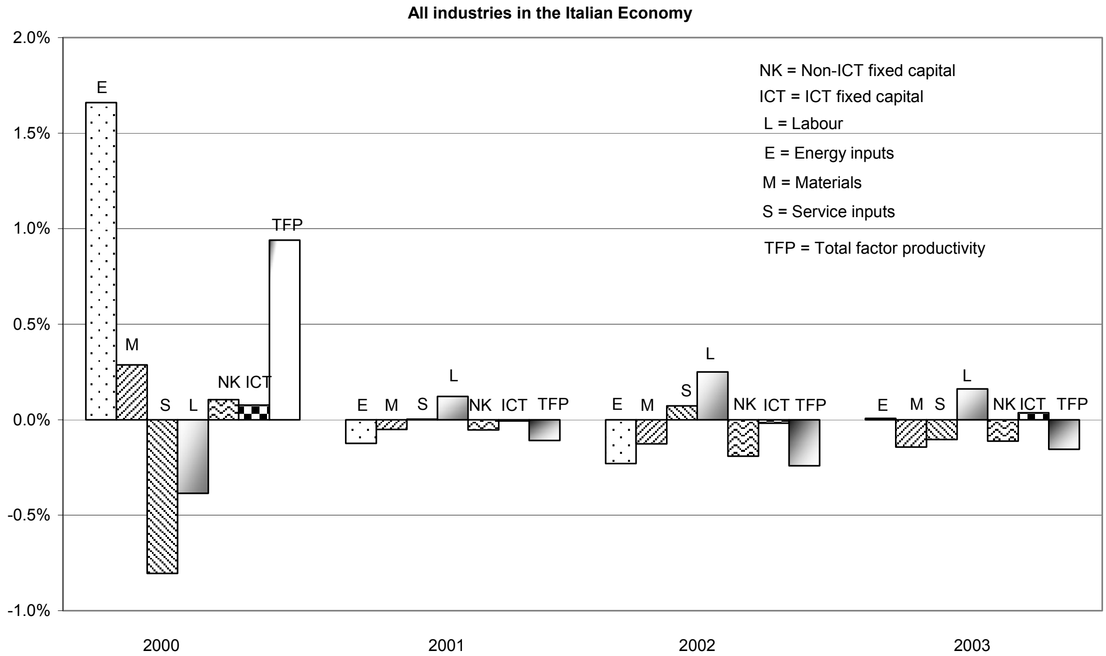


Figure 5. Measures of effects of TFP growth on real factor prices, based on the GL and KB profit functions



Appendix A. Different Types of Separability

The notion of separability was independently introduced, essentially in the same terms, by Leontief (1947a, 1947b) and Sono (1945, 1961), respectively.[15] They specified that two inputs x_i and x_j are separable from a third input x_k if and only if their marginal rates of substitutions are independent from the third input; that is, in our context,

(A-1) $$\frac{\partial}{\partial x_k}\left(\frac{T_i^t}{T_j^t}\right)=0$$

where $T_i^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial x_i]|_{T(\mathbf{y},\mathbf{x})=0}$ and $T_j^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial x_j]|_{T(\mathbf{y},\mathbf{x})=0}$. T_i^t and T_j^t denote, respectively, the marginal productivity of inputs x_i and x_j, evaluated with respect to the production efficiency frontier. This condition can be extended to the output space.

Inputs are separable from outputs and a function $f^t(\mathbf{x})$ exists, if and only if (A-1) applies to the derivatives of all ratios between marginal input productivities with respect to the single outputs (with x_k being replaced by y_k). Similarly, outputs are separable from inputs and a function $g^t(\mathbf{y})$ exists if and only if (A-1) applies to the derivatives of all ratios between marginal contributions of outputs with respect to the single inputs (with $T_i^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial y_i]|_{T(\mathbf{y},\mathbf{x})=0}$ and $T_j^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial y_j]|_{T(\mathbf{y},\mathbf{x})=0}$). Both outputs and inputs are mutually separable and the functions $f^t(\mathbf{x})$ and $g^t(\mathbf{y})$ exist within the transformation function if (A-1) applies, where x_k is replaced by any of the two functions $Y^t(\mathbf{y})$ and $X^t(\mathbf{x})$, while T_i^t and T_j^t are accordingly redefined with respect to the single inputs or outputs, respectively.

The technology of production is *input separable* when the separability conditions hold globally for all inputs and the transformation function can be written as

(A-2) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[\mathbf{y}, f^t(\mathbf{x})],$$

where the function $f^t(\mathbf{x})$ is a degree-one homogenous function (that is, if all the elements of $\mathbf{x}$ are multiplied by a scalar λ, then $f(\lambda\mathbf{x})=\lambda f(\mathbf{x})$).

Similarly, the technology of production is *output separable* when the separability conditions hold globally for all the outputs, in which case the transformation function can be defined as

(A-3) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[g^t(\mathbf{y}),\mathbf{x}]$$

where $g^t(\mathbf{y})$ is a degree-one homogenous function.

[15] Their definition refers to local separability (requiring differentiability of functions), but can be potentially extended to a concept of global separability. Stigum (1967), Gorman (1968), and Bliss (1975) have given other definitions of separability, which have been used extensively also by Blackorby, Primont and Russell (1978) in consumer theory.

We must note, at this point, that separability is a necessary but not a sufficient condition for constructing aggregates of inputs or outputs on the basis of the transformation function (A-2) or (A-3). Strotz (1959) and Gorman (1959) have shown that an aggregate of input quantities exists if in addition to these input quantities being separable, the resulting quantity index is a function homogeneous of degree one in its inputs (that is, if all these inputs change proportionally, also the quantity index changes by the same factor of proportionality). Green (1964, p. 25) has called these conditions "homogeneous functional separability."

The technology of production is *input-output separable* when the separability conditions hold globally and simultaneously for all the outputs and the inputs so that the transformation function can be defined as

(A-4) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[g^t(\mathbf{y}), f^t(\mathbf{x})],$$

where the internal structure of $f^t(\mathbf{x})$ is independent from that of $g^t(\mathbf{y})$. In this case, the vectors $\mathbf{y}$ and $\mathbf{x}$ are said to be mutually *weakly separable* within the internal structure of the transformation function.

Actually, it seems not appropriate, in this case, to distinguish between strong and weak separability since there are apparently only two groups of variables, $g^t(\mathbf{y})$ and $f^t(\mathbf{x})$. Following Lau (1972, pp. 282-287), however, $T(y,x)$ can be seen as an almost homogeneous function, which can be re-expressed in a linearly homogenous form since a third hidden argument can be explicitly considered:

$$T(\mathbf{y},\mathbf{x}) = \lambda^0 T(\mathbf{y},\mathbf{x}) = T(\lambda^k \mathbf{y}, \lambda \mathbf{x})$$

$$= \lambda^{-1} T[\lambda(\mathbf{x}+\mathbf{s}), \lambda x] = T^*(\mathbf{y},\mathbf{x},\mathbf{s}) = \lambda^{-1} T(\lambda \mathbf{x}, \lambda \mathbf{s}) = 0$$

where $s \equiv (\lambda^{k-1} x^{k-1} - 1)x$, which can be interpreted as a scale factor. Since there are more than two arguments, by applying Berndt and Christensen's (1973) theorems on weak and strong separability, we define the technology as *strongly input-output separable* with respect to the scale factor if $d(y_i / x_j)/dp_{sr} = 0$, where p_s is the reward to the scale factor. This case is that of constant returns to scale. If $d(y_i / x_j)/dp_{sr} \neq 0,$ then the technology can be defined as *weakly input-output separable*. This is the case of the example given above where the returns to scale are nonconstant.

The same reasoning applies to technical change.

A special case of the input-output separable technology implied by (A-4) is the *homothetically separable* technology that was defined by Shephard (1953, p. 43), which leads us, in our context, to the following general form of the transformation function:

(A-5) $$T^t(\mathbf{y},\mathbf{x}) \equiv g^t(\mathbf{y}) - F^t[f^t(\mathbf{x})]$$

$$= F^{t^{-1}}[g^t(\mathbf{y})] - f^t(\mathbf{x}) \qquad (\text{since } T^t(\mathbf{y},\mathbf{x}) = 0)$$

The transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ defined by (A-5) is a special case of (A-4), since it is based on the additional hypothesis that $f^t(\mathbf{x})$ is linearly homogeneous. It is worth noting, here, that, although $f^t(\mathbf{x})$ is an order-one homogeneous function, in general the *homothetic function* $F^t[f^t(\mathbf{x})]$ may fall into one of the broader classes of homogeneous and non-homogeneous functions[16]. Furthermore, for the output-input relation to be in "additive" form, the aggregates $g^t(\mathbf{y})$ and $f^t(\mathbf{x})]$ must be either mutually perfectly substitutable, or not at all substitutable. In this case, the technology is *additively separable* or *strongly separable* and the transformation function can be written as follows:

(A-6) $$T^t(\mathbf{y},\mathbf{x}) \equiv F^t[g(\mathbf{y}) - f(\mathbf{x})]$$

implying, by the implicit function theorem, $g(\mathbf{y}) = A^t[f(\mathbf{x})]$, which is the form that is usually assumed in the index number approach to productivity measurement. This relation requires, therefore, that the returns to scale are constant. If the separability holds globally as in (A-6), the vectors **y** and **x** are said to be mutually *strongly separable* within the internal structure of the transformation function.

Separability is a necessary but not a sufficient condition for constructing aggregates of outputs or inputs on the basis of the transformation function (A-4), since (non-linear) interaction effects between $f^t(\mathbf{x})$ and $g^t(\mathbf{y})$ could not be decomposable.

The separability conditions of outputs and inputs on the transformation function in the space of quantities can be translated in terms of separability conditions on cost, revenue, and net-profit functions. Berndt and Christensen (1973) have shown that the Leontief (1947) - Sono (1961) condition given by (A-1) can be translated in terms of the Allen-Uzawa partial elesticities of substitution. If (A-1) is true, then

(A-7) $$\frac{\partial}{\partial w_k}\left(\frac{C_i^t}{C_j^t}\right) = 0$$

where $C_i^t \equiv \partial C^t(\mathbf{w},\mathbf{y})/\partial w_i$, $C_j^t \equiv \partial C^t(\mathbf{w},\mathbf{y})/\partial w_j$, and w_k is the price of input x_k. By Shephard's lemma, $C_i^t = x_i$ and $C_j^t = x_j$. Therefore, we have

(A-8) $$\frac{\partial}{\partial w_k}\left(\frac{x_i}{x_j}\right) = 0$$

which means that the price of the k^{th} input does not affect the ratio between the two inputs x_i and x_j.

The condition (A-7) implies

(A-9) $$C_{ik}^t C_j^t = C_{jk}^t C_i^t$$

[16] Shephard (1953, pp. 41-50)(1970, pp. 30-36) has introduced the concept of the homothetic production function $F^t[(f^t(\mathbf{x})]$, which he defined to be a continuous, positive, monotone, increasing function of a homogeneous function of degree one. A homothetic production function is *non-homogeneous* if it changes by a factor that depends non-linearly on the scale of production when all inputs are multiplied by a positive scalar value λ, *i.e.* $F^t[(f^t(\lambda \cdot \mathbf{x})] = F^t[(\lambda \cdot f^t(\mathbf{x})] = \lambda^{\varphi(\mathbf{y})} \cdot F^t[(f^t(\mathbf{x})]$, whereas it is *homogeneous of degree r* if $F^t[(f^t(\lambda \cdot \mathbf{x})] = F^t[(\lambda \cdot f^t(\mathbf{x})] = \lambda^r \cdot F^t[(f^t(\mathbf{x})]$, with $r = 1$ meaning that it is *linearly homogeneous* and reflects constant returns to scale.

where $C_{ik}^t \equiv \partial C_i^t(\mathbf{w},\mathbf{y})/\partial w_k$. Multiplying both sides of (3.16) by $C^t/C_k^t \cdot C_i^t \cdot C_j^t$ yields

$$\text{(A-10)} \qquad \frac{C^t C_{ik}^t}{C_i^t C_k^t} = \frac{C^t C_{jk}^t}{C_j^t C_k^t}$$

where $C^t \equiv C^t(\mathbf{w},\mathbf{y})$. In other terms, the functional separability of inputs x_i and x_j from the third input x_k implies $\sigma_{ik} = \sigma_{jk}$, where $\sigma_{ik} \equiv C^t C_{ik}^t / C_i^t C_k^t$ and $\sigma_{jk} \equiv C^t C_{jk}^t / C_j^t C_k^t$ are the Allen-Uzawa partial elasticities of substitution between the pairs of inputs x_i and x_k and between the pairs of inputs x_j and x_k.

The following cases of input and/or output separability can be distinguished:

Input separability. In the case the separability condition is referred to a single pair of inputs x_i and x_j with respect to a single output y_k, (A-7) is replaced by

$$\text{(A-11)} \qquad \frac{\partial}{\partial p_k}\left(\frac{C_i^t}{C_j^t}\right) = 0$$

which is equivalent to

$$\text{(A-12)} \qquad \frac{\partial}{\partial p_k}\left(\frac{x_i}{x_j}\right) = 0$$

Similar conditions can be derived for the conditional revenue functions and the Marshallian and Paretian conditional net-profit functions. In terms of the general conditional net-profit function, the separability condition corresponding to (A-8) becomes:

$$\text{(A-13)} \qquad \frac{\partial}{\partial q_k}\left(\frac{\Pi_i^{McF\,t}}{\Pi_j^{McF\,t}}\right) = 0,$$

where $\Pi_i^{McF\,t} \equiv \partial \Pi^{McF\,t}(\mathbf{q},\mathbf{k})/\partial q_i$.

If the technology is globally input separable as in (A-2), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ can be written as $T^t[\mathbf{y}, f^t(\mathbf{x})] = 0$, where the separability conditions stated above are valid for any pair of inputs in **x** with respect to every single output in **y**, then the dual profit, cost, and revenue functions can be written as follows:

$$\text{(A-14)} \qquad \Pi^{McF\,t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF\,t}(\mathbf{q}, K(\mathbf{k}))$$

$$\text{(A-15)} \qquad \Pi^t(\mathbf{p},\mathbf{w}) \equiv \Pi^t[\mathbf{p}, \omega(\mathbf{w})]$$

$$\text{(A-16)} \qquad C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\omega(\mathbf{w}), \mathbf{y}]$$

$$\text{(A-17)} \qquad R^t(\mathbf{p},\mathbf{x}) \equiv R^t(\mathbf{p}, f(\mathbf{x})).$$

Output separability. If the technology is globally output separable as in (A-3), so that so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t[g^t(\mathbf{y}),\mathbf{x}]=0$, then the dual profit, cost, and revenue functions can be written as (see also Lau, 1978, p. 175):

(A-18) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF^t}[\phi(\mathbf{q}),\mathbf{k}]$$

(A-19) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \Pi^t[\varphi(\mathbf{p}),\mathbf{w}]$$

(A-20) $$C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\mathbf{w},g(\mathbf{y})]$$

(A-21) $$R^t(\mathbf{p},\mathbf{x}) \equiv R^t[\varphi(\mathbf{p}),\mathbf{x}].$$

Input-output (weak) separability. If the technology is globally input-output separable as in (A-4), so that so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t[g^t(\mathbf{y}),\mathbf{x}]=0$, then the dual profit, cost, and revenue functions can be rewritten as (see also McFadden, 1978, p. 58):

(A-22) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF^t}[\phi(\mathbf{q}),K(\mathbf{k})]$$

(A-23) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \Pi^t[\varphi(\mathbf{p}),\omega(\mathbf{w})]$$

(A-24) $$C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\omega(\mathbf{w}),g(\mathbf{y})]$$

(A-25) $$R^t(\mathbf{p},\mathbf{x}) \equiv R^t[\varphi(\mathbf{p}),f(\mathbf{x})]$$

Homothetic input-output separability. If the technology is homothetically separable as in (A-5), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t(\mathbf{y},\mathbf{x}) \equiv g^t(\mathbf{y}) - F^t[f^t(\mathbf{x})] = 0$ or, equivalently, $T^t(\mathbf{y},\mathbf{x}) \equiv F^{t^{-1}}[g^t(\mathbf{y})] - f^t(\mathbf{x}) = 0$ where f^t being linearly homogenous functions and f^t being a homothetic function, then the dual profit, profit, and revenue functions can be rewritten as follows (see Shephard, 1953, 1970 for the case of the cost function, and McFadden, 1978, p. 58, Denny and Pinto, 1978, p. 253, and Lau, 1978, pp. 159-160):

(A-26) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \phi^t(\mathbf{q}) \cdot F^t[K^t(\mathbf{k})],$$

(A-27) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv [\pi^t(\mathbf{p}) - \pi^t(\mathbf{w})],$$

(A-28) $$C^t(\mathbf{w},\mathbf{y}) \equiv \omega^t(\mathbf{w}) \cdot F^{t^{-1}}[g(\mathbf{y})], \text{ and}$$

(A-29) $$R^t(\mathbf{p},\mathbf{x}) \equiv \varphi^t(\mathbf{p}) \cdot F^t[f(\mathbf{x})],$$

where the function F^t $(\cdot)$ was defined in (A-5). Note that $f(\mathbf{x}) = F^{t^{-1}}[g^t(\mathbf{y})]$ and $g^t(\mathbf{y}) = F^t[f^t(\mathbf{x})]$. Therefore, as it is expected, $C^t(\mathbf{w},\mathbf{y}) = \omega^t(\mathbf{w}) \cdot f^t(\mathbf{x})$ and $R^t(\mathbf{p},\mathbf{x}) = \varphi^t(\mathbf{p}) \cdot g^t(\mathbf{y})$.

In the homothetic case, not only the internal structure, but also the levels of $\varphi^t(\mathbf{p})$ and $f^t(\mathbf{x})$ are mutually independent, as well as those of $\omega^t(\mathbf{w})$ and $g^t(\mathbf{y})$. Homothetic separability can be considered as a special case of input-output separability.

If the technology is characterized by additive (strong) input-output separability as in (A-6), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t(\mathbf{y},\mathbf{x})\equiv F^t[g(\mathbf{y})-f(\mathbf{x})]=0$ (implying, by the implicit function theorem, $g(\mathbf{y})=A^t[f(\mathbf{x})]$, then the dual profit, cost, and revenue functions can be written in a way directly derivable from the homothetic separability case, where the function $F^t[f^t(\mathbf{x})]$ reduces to $f^t(\mathbf{x})$, and $F^{t^{-1}}[g(\mathbf{y})]$ to $g(\mathbf{y})$.

Separability of technical change from outputs and/or inputs can be studied in a way analogous to that for input and output separability. If technical change of the transformation function is Hicks-neutral, then it is homothetically separable from input and output changes. The dual cost and revenue can be written as follows:

(A-30) $$C^t(\mathbf{w},\mathbf{y})\equiv\omega(\mathbf{w})\cdot A^{t^{-1}}\cdot F^{-1}[g^t(\mathbf{y})] \text{ and}$$

(A-31) $$R^t(\mathbf{p},\mathbf{x})\equiv\varphi(\mathbf{p})\cdot A^t\cdot F[f(\mathbf{x})]\cdot$$

Note that $f(\mathbf{x})=A^{t^{-1}}\cdot F^{-1}[g(\mathbf{y})]$ and $g(\mathbf{y})=A^t\cdot F[f(\mathbf{x})]$. Therefore,

(A-32) $$C^t(\mathbf{w},\mathbf{y}))=\omega(\mathbf{w})\cdot A^{t^{-1}}\cdot F^{-1}[g(\mathbf{y})]=\omega(\mathbf{w})\cdot f(\mathbf{x}) \text{ and}$$

(A-33) $$R^t(\mathbf{p},\mathbf{x})=\varphi(\mathbf{p})\cdot A^t\cdot F[f(\mathbf{x})=\varphi(\mathbf{p})\cdot g(\mathbf{y}).$$

In the case of constant returns to scale and perfect competition, the long-run equilibrium yields

$$\frac{R^t(\mathbf{p},\mathbf{x})}{C^t(\mathbf{w},\mathbf{y})}=\frac{\varphi(\mathbf{p})\cdot g(\mathbf{y})}{\omega(\mathbf{w})\cdot f(\mathbf{x})}=1 \text{ or, equivalently, } \frac{g(\mathbf{y})}{f(\mathbf{x})}=\frac{\omega(\mathbf{w})}{\varphi(\mathbf{p})}=TFP.$$

Appendix B   The Properties Index Numbers

When the reference inputs that are conditional for the revenue function are not consistent with the reference outputs in the cost function, the revenue-based index numbers generally differ in value from the cost-based index numbers.

As anticipated during the description of cost-based output quantity indexes, the separability conditions described in section 2 are of great importance for the resulting index numbers to correctly represent the changes that they are intended to measure. We emphasize here that, if strong input-output separability (constant returns to scale) is not present, unless a more general formulation is found, it could not be possible to find a uniquely determined index number. In case of input-output weak separability, in fact, the construction of index numbers can be possible, but under additional arbitrary hypotheses concerning the degree of input-output substitutions.

More specifically, under constant returns to scale, economic index numbers are invariant with respect to the reference variables. This means that price indexes are functions only of elementary prices and quantity indexes are functions only of elementary quantities. Therefore, the candidate index numbers presented in Table 1, should be rewritten in canonical form with no reference variables. Samuelson and Swamy (1974, pp. 571-572) have established the "completeness theorem" for economic canonical index numbers for the homothetic case, stating that these index numbers (independently from their functional form) satisfy all the test criteria of Fisher (1911) appropriate to the primitive one-good case. These test criteria are the following:

General mean of price relatives (or linear homogeneity of the price index) test: If all the elementary prices are multiplied by a λ, also the resulting price index number is multiplied by a λ.

Time-reversal test: The index becomes the inverse of itself if the time order is reversed.

Circular-reversal (or transitivity) test: The index number comparing two situations does not change if it is constructed transitively by chaining index numbers referring to a third observation point.

Dimensional invariancy test: The index is invariant with respect to the dimensional change of the variables.

Factor-reversal test: The price index multiplied by the quantity index equals the total nominal-value index. This test is called "strong factor-reversal test" if the price and quantity index numbers have the same functional form and "weak factor-reversal test" if the two index numbers have different functional forms.

Samuelson and Swamy (1974, p. 575), in the context of economic index numbers, drop the strong factor-reversal test in favor of the weak factor-factor reversal test. They write that (with notation adjusted): "We must stress again that the factor-reversal test offers no stumbling block for our definitions of $P(\mathbf{p}^0,\mathbf{p}^1;\overline{\mathbf{q}}^r)$ and $Q(\mathbf{q}^0,\mathbf{q}^1;\overline{\mathbf{p}}^r)$ if, as we should do logically, we drop the *strong* requirement that the *same* formula should apply to $Q(\mathbf{q})$ as to $P(\mathbf{p})$." They then jokingly add: "A man and wife should be properly matched; but that does not mean I should marry my identical twin!"

As for the circularity test, rather surprisingly, even in the non-homothetic case, the economic index numbers do not fail to satisfy it. This is due to the fact that, differently from the traditional index numbers that have to rely on the information limited only to prices and quantities, economic index numbers can be constructed using *also* an explicit (known) functional form of the underlying aggregator function. This incorporates the additional information concerning technology-related behavioral choices. At least in principle, we could calculate the economic index numbers by simulation of the value function at hand at the given prices *and* reference variables and, therefore compare the results obtained (which are conditional to the *same* reference variables) transitively between any different situations whatsoever. With this in mind, Samuelson and Swamy (1974, p. 575) see Fisher's problem from the "external" economic point of view with subtle irony: "Where most of the older writers balk, however, is at the circular test that free us from one base year. Indeed, so enamoured did Fisher become with his so-called Ideal index […] that, when he discovered it failed the circular test, he had the hubris to declare '…, therefore a *perfect* fulfillment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous' (1922, p. 271). Alas Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning".

Most important, both the price and quantity index number satisfy the linear homogeneity test (*i*) in the homothetic case. Samuelson and Swamy (1974, pp. 576-577) adjoined to this test the requirement that both price and quantity index numbers are homogeneous of degree zero with respect to the weights (that is, they are not affected by the scale of the weights). In other words, this widened test, which they called "widened (*i*)", requires that the economic price and quantity indexes are to be homogeneous of degree one in the elementary prices and quantities, respectively, and homogeneous of degree zero in their respective weights. This widened test is satisfied in the homothetic case, but fails in the non-homothetic case while satisfying all the other tests (*ii*)-(*v*). It is consistent with the homothetic separability requirements on the underlying economic function for aggregation. In particular, in the general non-homothetic case, if an economic index of prices (quantities) always fulfils, by construction, the requirements of the linear homogeneity test, its dual quantity (price) index number that is constructed implicitly by deflating the underlying nominal value function by the primal price (quantity) economic index fails to pass this test. Moreover, in the non-homothetic case, the direct economic index fails to be homogeneous of degree zero with respect to the variables taken as weights, while the implicit dual index is always homogeneous of degree zero but depends also on the reference variables.

As a consequence of these theoretical results, if the price *and* quantity index numbers are constructed independently so that they both satisfy the linear homogeneity test (*i*) (this is the procedure proposed by Pollak, 1971 and Diewert, 1983), then the factor-reversal test will fail in the non-homothetic case. More specifically, in the non-homothetic case, the economic price (quantity) index number can still satisfy the linear homogeneity property (*i*), but will fail the Samuelson-Swamy adjoined requirement that they are to be also invariant with respect to the reference quantities (prices). Quoting their words (and using our notation), "[i]f Fisher had adjoined to (i*) the requirement that the quantity index is never to be affected by scale changes in $\mathbf{p}^1$ and $\mathbf{p}^0$ (which leave their 'weightings' unchanged), we'd learn in the nonhomothetic case that both indexes must fail this widened (*i**) test" (p. 577).

The test for zero-degree homogeneity (invariance) of the price index with respect to the *current-period* quantity weights has been attributed to Vogt (1980, p. 70) by Diewert (1992, p. 217), who, in turn, proposed the test for the zero-degree homogeneity (invariance) of the price index with respect to the *base-period* quantity weights. Diewert (1992, p. 217, fn. 9) himself, however, reminds us that Irving Fisher (1911, pp. 400-406), in his "almost forgotten (but nonetheless brilliant work)" had actually considered the linear homogeneity requirements of the *proportionality test* together with the requirements of the zero-degree homogeneity of the price indexes with respect to the current-period quantity weights (which we may note turns to be the counterpart of the widened (*i**) test devised by Samuelson-Swamy for economic index numbers[17]). He considered this test as the most important among the eight tests that he had devised for price indexes because it might indicate what type of quantity weights was required. However, the later Fisher (1922, pp. 420-421) no longer seemed to consider this test was important and reduced the relevant tests to the five referred to above. It is remarkable that Samuelson and Swamy (1974, pp. 576-577) have, instead, considered this test among the most important and critical tests that should be satisfied by an economic index number.

[17] We must, however, consider the difference between the index number formulas and the economic index numbers. Fisher (1911)(1922) considered the base- or current-period quantities or prices as reference variables in his formulas, whereas Samuelson-Swamy (1974, pp. 567-68) considered a *variable* optimal basket of price-dependent quantities required to attain a reference level of living or output in defining their economic price index and a given reference price situation in defining their economic quantity index. (In fact, they defined this economic quantity index as that obtained implicitly by deflating the nominal expenditure by means of the economic price index.)

Appendix C The Special Importance of Homogeneity

Because of the duality relationship between the price and quantity indexes, the degree-one (linear) homogeneity of both these two aggregating indexes of economic variables is a sufficient and necessary condition for their dual counterparts to be of degree-zero homogeneous in the weights of aggregation. At the same time, the degree-zero homogeneity in the weights of aggregation of both economic indexes is a sufficient and necessary condition for their dual counterparts to be degree-one homogeneous in their economic variables. If an index number fails to be zero-degree homogeneous in the weights, then it is non-invariant with respect to the reference variables. These reference variables are consequently non-separable from the variables to be aggregated. Under this condition, aggregation is not possible and the index number itself loses its economic meaning[18].

In the economic approach, economic index numbers are usually based on cost and revenue functions, which are always linearly homogeneous in prices. This means that, in this approach, the linear homogeneity property of quantity indexes and the invariance of price index numbers are put in question.

Archibald (1977, p.70), showed that, while the revenue function and the output-price indexes are always homogeneous of degree one in the output prices (by construction), the output quantity index obtained implicitly by deflating the nominal revenue by means of the revenue-based economic price index does not satisfy the requirements of the linear homogeneity test in the general non-homothetic case.

Similarly, Fisher (1988) noted that, while the cost function and the Konüs-type input-price indexes are always homogeneous of degree one in the input prices (by construction), the Paasche-Allen-type index number of real aggregate inputs is not, in the general non-homothetic case, homogeneous of degree one in the input quantities.

Under the assumption of cost minimization, the economic theory of index numbers implies that

(C-1) $$C^0(\mathbf{w}^0,\mathbf{y}^0) = \mathbf{w}^0 \cdot \mathbf{x}^0$$

(C-2) $$C^1(\mathbf{w}^1,\mathbf{y}^1) = \mathbf{w}^1 \cdot \mathbf{x}^1$$

(C-3) $$C^0(\mathbf{w}^1,\mathbf{y}^0) \le \mathbf{w}^1 \cdot \mathbf{x}^0$$

(C-4) $$C^1(\mathbf{w}^0,\mathbf{y}^1) \le \mathbf{w}^0 \cdot \mathbf{x}^1$$

If $\mathbf{x}^1 = \lambda \cdot \mathbf{x}^0$, where λ is a positive scalar, then using (C-3),

(C-5) $$[C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^1,\mathbf{y}^0)] \ge \mathbf{w}^1 \cdot \lambda \cdot \mathbf{x}^0 / \mathbf{w}^1 \cdot \mathbf{x}^0 = \lambda,$$

We can also show that the Laspeyres-Allen-type index number of real aggregate inputs is not, in the general non-homothetic case, homogeneous of degree one in the input quantities. That is, if $\mathbf{x}^1 = \lambda \cdot \mathbf{x}^0$, then, using (C-4), we have

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[18] This point, highlighted by Samuelson and Swamy (1974, pp. 576-577), is a reformulation of the theory of separability and aggregation set up by Strotz (1959) and Gorman (1959).

(C-6) $$[C^1(\mathbf{w}^0,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0)] \leq \mathbf{w}^0 \cdot \lambda \cdot \mathbf{x}^0 / \mathbf{w}^0 \cdot \mathbf{x}^0 = \lambda .$$

The results just obtained, suggest that, if it is possible to find a particular reference input-price vector $\mathbf{w}*$ such as

(C-7) $$[C^1(\mathbf{w}*,\mathbf{y}^1)/C^0(\mathbf{w}*,\mathbf{y}^0)] = \mathbf{w}*\cdot\lambda\cdot\mathbf{x}^0 / \mathbf{w}*\cdot\mathbf{x}^0 = \lambda$$

then an economic input-quantity index that is *locally* linearly homogeneous can be found. However, also this particular economic index number would not be satisfactory, since the dual (implicitly derivable) input-price index, which is to be consistent with the (weak) factor-reversal test, would not satisfy, in general, the Fisher-Samuelson-Swamy extended homogeneity property. To see this, let us consider the following implicit input price index number obtained as the ratio between the nominal-value index of total costs and a linearly homogeneous input-quantity index number, independently constructed with reference to the price vector $\mathbf{w}*$:

(C-8) $$\widetilde{W}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}*) \equiv \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \cdot \frac{1}{X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}*)}$$

In contrast with the economic input-price index numbers, this index fails, in general, the linear-homogeneity test. If all the elementary input prices change by a scalar λ between period 0 and period 1, then the Paasche- and Laspeyers-weighted economic index numbers change by the same factor λ

(C-9) $$\frac{C^1(\lambda\cdot\mathbf{w}^0,\mathbf{y}^1)}{C^1(\mathbf{w}^0,\mathbf{y}^1)} = \frac{C^0(\lambda\cdot\mathbf{w}^0,\mathbf{y}^0)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \lambda$$

whereas, in general, from (C-8) we have

(C-10) $$\widetilde{W}_C(\mathbf{w}^0,(\lambda\cdot\mathbf{w}^0),\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}*) \equiv \frac{C^1(\lambda\cdot\mathbf{w}^0,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \cdot \frac{1}{X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}*)} \neq \lambda$$

since

(C-11) $$C^1(\mathbf{w}^0,\mathbf{y}^1) = \mathbf{w}^0 \cdot \mathbf{x}^1(\mathbf{w}^0,\mathbf{y}^1)$$
$$\neq C^0(\mathbf{w}^0,\mathbf{y}^0)\cdot X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}*) .$$

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