Functional Forms for Age-Efficiency Functions

The literature on capital deterioration has highlighted the straightline, one-hoss shay, and geometric forms, which are pedagogically useful and have simple age-price profiles. Tests of which form best describes actual age-related efficiency losses are hampered, however, for want of an overall form that subsumes all three as limiting cases and has a convenient age-price function. The leading research in the field in fact begins with a flexible age-price function, from which the corresponding efficiency form can be derived ex post [Hulten and Wykoff, 1981a and 1981b], but that approach forces the discount rate into the efficiency function. One would like, then, to “work up” from the efficiency side: to cover straightline, one-hoss shay, and geometric forms as special cases of something broader, and to explore a still wider range of efficiency shapes in a concise way. The good news here is that there is a two-parameter age-efficiency profile that specializes to the three familiar forms and has a simple price function. That profile in turn is a special case of one of four different three-parameter functions that allow “backward-S” or “reclining-chair” shapes and that may be seen as second-order approximations to any primal form. The bad news is that the three-parameter forms do not have simple price-side duals, which must therefore be evaluated numerically.

To begin to see this, consider the differential equation describing the efficiency, \( E(s) \), of a machine of age \( s \), from \( E(0)=1 \) down to \( E(L)=0 \):

\[
E'(s) = AE(s)^2 + BE(s) + C \leq 0, \quad (1)
\]

where \( A, B, \) and \( C \) are real parameters. Boundary and equilibrium conditions imply:

\[
E'(0) = A + B + C \leq 0 \quad (2)
\]

\[
E'(L) = C \leq 0
\]

\[
E'(s_{eq}) = 0 \Rightarrow E(s_{eq}) = \left[-B \pm (B^2 - 4AC)^{1/2}\right]/(2A). \quad (3)
\]

The equilibrium condition is not formally necessary, since efficiency never stops decaying in the relevant range. Now, choosing \( B^2 - 4AC \equiv d^2 \geq 0 \) to avoid oscillations, solve (1) subject to the boundary conditions to find:

\[
E(s) = \frac{\left[e^{ds} - e^{dl}\right]/(1-e^{dl})}{\left[d(1+e^{dl}) - B(1-e^{dl})\right]/[d(e^{ds}+e^{dl}) - B(e^{ds} - e^{dl})]]. \quad (3)
\]

To guarantee \( E'(s) \leq 0 \), impose \( B \geq d(1+e^{dl})/(1-e^{dl}) \), which is easy to write but difficult to apply. A reparametrization that is not obvious but that is convenient and nonrestrictive\(^1\) is:

\[
B = d(1+e^{dl})/(1-e^{dl}) - 2d(e^{dl/2})/(1-e^{dl}). \quad (4)
\]

Substitute for \( B \) in (3) according to (4), then put \( d=a/L \) to phrase age as a fraction of the lifespan. The resulting form, which is indifferent to the sign of \( a \), completes the basic development:

\[
E(s) = (e^{asL} - e^a)/(e^{asL} - e^a) + e^{b+2aL}(1-e^{asL}). \quad (5)
\]

\(^1\) Parameters \( B \) and \( b \) are in a one-to-one relation in (4). Note particularly that: \( \lim_{b\to-\infty} B \to +\infty \), \( \lim_{b\to+\infty} B = \mp d \), \( \lim_{b\to-\infty} B = d(1-e^{dl/2})/(1+e^{dl/2}) \), and \( \lim_{b\to+\infty} B = d(1+e^{dl})/(1-e^{dl})<0 \).
The preceding transformations leave:
\[A = a(e^{ad^2} - e^b)(e^{ad^2} - e^b)/(e^d - 1)/L \quad B = ae^{ad^2}(e^{ad^2} + e^{-ad^2} - 2e^b)/((1 - e^d)/L) \quad C = ae^{ad^2}/(1 - e^d)e^b/L. \]  

Positive L and real a and b jointly imply \( C \leq 0 \) and \( A + B + C \leq 0 \), so the preliminary conditions are indeed met. Further, “equilibrium efficiency”—which would occur, depending on the sign of \( a \), when remotely early or remotely late \( s \) drives \( e^{asL} \) to zero—is \( e^{ad^2}/(e^{2ad^2} - e^b) \). Also, substituting (6) into (1) and rearranging yields:

\[E'(s) = [ae^{ad^2}/(1 - e^d)e^b/L] [1 - (1 - e^{b-ad^2}) E(s)]^2 + (a/L) E(s) [1 - (1 - e^{b-ad^2}) E(s)]. \]  

so the constrained differential equation may be thought of as a weighted sum of the two simpler problems in bold type. The first is solved by \( E(s) = [L - s]/[L - (1 - e^d)s] \) as \( a \to 0 \), which nullifies the second problem and forces \( C = -e^b/L. \) The second is solved by \( E(s) = e^{ds}/[e^{ds} + e^{b/(1-e^d)}] \) as \( L \to \infty \) (after first substituting \( b \to b - al/2 \) and \( a \to dL \), and choosing \( d < 0 \), which nullifies the first problem. The respective sub-solutions are recognizable as the Faucett-OPT hyperbolic form and the logistic curve of demographic studies. Setting \( b = \bar{b} = al/2 \) in the combined problem yields \( E(s) = (e^{aol} - e^{ta})/(1 - e^{ta}) \), which is not indifferent to the sign of \( a \); setting \( b = 0 \) in the first problem alone (i.e. given \( a \to 0 \)) gives the straightline \( E(s) = 1 - s/L \); setting \( b = -al/2 \) in the second problem alone gives the geometric \( E(s) = e^b \) (again, as \( L \to \infty \), where \( a = dL \) iff \( d < 0 \)). In general, \( b \) is a “bias” term, governing where (or whether) the inflection point \( s^* \) falls as a fraction of the lifespan:

\[
\text{If } b = \ldots, \quad -|a/2| < \ln[2e^{ad^2}/(1 + e^d)] < 0 < \ln[(1 + e^d)/(2e^{ad^2})] < \left|a/2\right|, \quad \text{then } s^*/L = \ldots \quad \infty > 1 > \frac{1}{2} > 0 > -\infty.\]

Values of \( b \) outside the \([-|a/2|, |a/2|]\) interval imply a complex inflection point. Yet in this forest of forms I have found that only the specialization \( b = \bar{b} = al/2 \), which contains the geometric, straightline, and one-hoss shay shapes as special cases, has a simple age-price profile.

To express (5) in terms of trigonometric functions, divide numerator and denominator by \( 2e^{(1+s)L/2} \), then rearrange to find:

\[E(s) = \sinh[(al/2)(1 - s/L)]/[\sinh[(al/2)(1 - s/L)] + e^b\sinh[(al/2)(s/L)]]. \]  

which inherits all the properties of its non-trigonometric predecessor.  

The specifications discussed so far describe “one-hump” and “backward-S” curves well, but not “reclining-chair” shapes. One way to get these is to reflect the standard forms across the “45° line”—i.e., swap “E” and “s/L” in (5) or (9), then solve for E. Two representations of the reflected forms are:

\[E(s) = \ln[|(s/L)e^b + (1 - s/L)e^{ad^2}|/(s/L)e^b + (1 - s/L)e^{ad^2})]/a \quad \text{or}\]

\[= 2\arctanh\{1-(s/L)\sinh(al/2)|/(s/L)e^b + (1-s/L)cosh(al/2)|]/a. \]

Only the Faucett-OPT hyperbolic form is self-reflective.  

\[\text{If there is a real inflection point, it will occur where } e^{asL} = (e^{ad^2} - e^b)/(e^b - e^{ad^2}). \text{ Solve for } e^b \text{ and plug into (5) for the inflection efficiency: } E(s^*) = \tanh[(al/2)(1 - s^*/L)]/[\tanh[(al/2)(1 - s^*/L)] + \tanh[(al/2)(s^*/L)]]. \]  

The upshot is that for \( 0 \leq s^* \leq L \), inflection points occur only in the two regions between \( E = 1 - s/L \) and \( E = \frac{1}{2} \).
Another way to access “reclining-chair” shapes is to confront the oscillatory solution—i.e., where the real parameters $B^2 - 4AC \equiv -\delta^2 \leq 0$. The constrained solution formally resembles (3):

$$E(s) = [(e^{i\delta} - e^{i\delta/2})/(1-e^{i\delta/2})] [i\delta(1+e^{i\delta}) - B(1-e^{i\delta/2})] / [i\delta(e^{i\delta} + e^{i\delta/2}) - B(e^{i\delta} - e^{i\delta/2})],$$

(11)

and requires $B \geq i\delta(1+e^{i\delta})/(1-e^{i\delta})$ to guarantee a downward slope. To keep $B$ real, apply:

$$B = i\delta(1+e^{i\delta})/(1-e^{i\delta}) - 2i\delta e^{i\delta/2}/(1-e^{i\delta}) = \delta e^{i\beta} \text{Csc}(\delta L/2) - \text{Cot}(\delta L/2).$$

(12)

to (11), to find:

$$E(s) = (e^{\alpha s}-\alpha)/(e^{\alpha s}-\alpha - 1)/L$$

(13)

and 2

$$= \sin((\alpha/2)/(1-s/L)) / \{(\sin((\alpha/2)(1-s/L)) + e^{\beta}\sin((\alpha/2)(s/L))\},$$

where $\alpha = \delta L$ must lie between $-2\pi$ and $2\pi$ for continuity (although the sign of $\alpha$ doesn’t matter). Again as in the non-oscillatory solution, the parameters of the original problem—all still real—work out as:

$$A = i\alpha(e^{i\alpha/2} - e^{-i\alpha/2})(e^{i\alpha/2} - e^{-i\alpha/2} - 1)/L$$

(14)

$$B = i\alpha e^{i\alpha/2}((e^{i\alpha/2} - e^{-i\alpha/2} - 1)/e^{i\alpha/2})/L$$

$$C = \delta e^{i\beta} \text{Csc}(\alpha/2) - \text{Cot}(\alpha/2)/L$$

$$= \text{Csc}(\alpha/2)/L,$$

so the equilibrium solution is $e^{i\alpha s}/(e^{i\alpha s} - e^-\beta)$. However, stating the oscillatory problem as a weighted sum of two simpler problems, as in (7) but with $i\alpha$ now replacing $a$ and $\beta$ replacing $b$, is less useful because real $\beta$ cannot equal $\pm i\alpha/2$. In fact I have encountered no specializations of the oscillatory solution with simple closed-form age-price profiles. It is also more difficult to use $\beta$ to gauge the bias of the inflection point in (13) than it was to use $b$ in (5) and (9):^4

$$\text{If } \beta = \ldots \rightarrow \infty < \ln[\cos(\alpha/2)] < 0 < \ln[\sec(\alpha/2)] < \infty \text{ if } -\pi < \alpha < 0$$

(15)

then $s*/L = \ldots 1 - |\pi/\alpha| \leq 0 < \frac{1}{2} < 1 \leq |\pi/\alpha|.$

To allow “backward-S” curves in the oscillatory context, reflect (13) across the “45° line,” to find:

$$E(s) = \ln\left\{ [(s/L)e^{\beta} + (1-s/L)e^{-i\alpha/2}]/[(s/L)e^{\beta} + (1-s/L)e^{-i\alpha/2}]/(i\alpha)\right\}$$

(16)

$$= 1 - \pi/|\alpha| + 2 \text{ArcTan}\{[(1-s/L)e^{\beta} + (s/L)e^{-i\alpha/2}]/[(s/L)e^{\beta} + (s/L)e^{-i\alpha/2}]/(s/L)\sin(\alpha/2)]/\alpha.$$  \(\ldots^5\)

The notational similarities between (5) or (9) and (13) and between (10) and (16) invite comparisons, but neat correspondences are most unlikely because the real and oscillatory solutions describe different families of curves. One equivalence is exact:

^3 Real $B$ and $\beta$ are one-to-one in (12). Note: $\lim_{B \rightarrow \infty} (S/L) \leq 0$, $\lim_{B \rightarrow \infty} (1+S/L) \leq 0$, $\lim_{B \rightarrow \infty} (1+S/L) = -\delta \text{Cot}(\delta L/4)$, and $\lim_{B \rightarrow \infty} (1+S/L) = -\delta \text{Cot}(\delta L/2)$. Insisting on real $\beta$ prevents $\lim_{B \rightarrow \infty} B = \mp i\delta$.

^4 Analogously with note 2, if there is a real inflection point, it will occur where $e^{|\alpha s/2|} = (e^{i\alpha/2} - e^{-i\alpha/2})^2/e^{-i\alpha/2}$, i.e., where $s*/L = -(2\alpha)\text{ArcTan}(\cos(\alpha/2)e^{\beta} - \alpha)/e^{\beta} - e^{-2\beta}$). Next, solve for real $e^\beta$ and plug into (13) for the oscillatory inflection efficiency: $E(s*) = \text{Arctan}(\alpha/2)(1-s*/L) + \text{Arctan}(\alpha/2)(s*/L)$. Permitted values for $s*/L$ then occur between $|\pi/\alpha|$ and $-|\pi/\alpha|$, implying that for $0 \leq s \leq s*/L$ and $-2 \pi \leq s \leq 2 \pi$, inflection points occur in the two regions between $E = 1-s/L$ and $s = L/2$. Additionally, the limiting values for $\beta$ in (15) are $\pm \infty$, the expression $e^\beta$ is very efficient: $\beta \pm \infty$ is “close enough” to $\beta \rightarrow \pm \infty$.

^5 The analog of (10) would be $E(s) = 2 \text{ArcTan}\{[(1-s/L)\sin(\alpha/2)/(s/L)e^{\beta} + (1-s/L)\cos(\alpha/2)]/\alpha$, which unfortunately jumps whenever $s/L = \cos(\alpha/2)/(\cos(\alpha/2) - e^{\beta})$, so any $|\alpha/\pi|$ would force a discontinuity at an interior $s$ (i.e., $0 \leq s \leq L$). The form in the text is continuous for all $s > 0$, even for putative $s > L$. 
\[
\lim_{\alpha \to 0} \frac{\sin[(\alpha/2)(1-s/L)]}{[\sin[(\alpha/2)(1-s/L)] + e^\beta \sin[(\alpha/2)(s/L)]]} = \frac{[L-s]}{[L-(1-e^\beta)s]},
\]
so the hyperbolic form stands at the border of the real and oscillatory solution concepts. Nonetheless, it is possible by statistical techniques to find curves from the two families that are quite close. For example, if we use the non-oscillatory form (5) and its reflection (10), with parameters \(a=10, b=-2,\) and \(L=20,\) to generate two sets of a hundred “observations” on efficiency, both evenly spaced by age from \(s=0.2\) through \(s=20,\) the best nonlinear least-squares fit of form (16) finds \(\alpha=5.11842\) and \(\beta=-0.851324,\) while the best nonlinear least-squares fit for (13) is \(\alpha=5.31796\) and \(\beta=-0.865143.\) A plot of the curves shows them crossing several times:

We finish with four different curves, expressed as either exponential or trigonometric functions, that trace “one-hump” [e.g., (5) or (9), (10), (13), and (16)], “backward-S” [e.g., (5) or (9), and (16)], and “reclining-chair” [e.g., (10) and (13)] age-efficiency schedules. Two of the four derive from the non-oscillatory solution (i.e., such that real \(B^2 \geq 4AC\)) to the original differential equation [e.g., (5) or (9) and the reflected form (10)]; two flow from the oscillatory solution (i.e., such that real \(B^2 \leq 4AC\)) [e.g., (13) and its reflection (16)]. Under both solution concepts, the only binding restrictions are that the original parameters \(A, B,\) and \(C\) stay real and that efficiency decline monotonically from \(E(0)=1\) down to \(E(L)=0,\) so the four forms are likely decent approximations to any nonincreasing normalized age-efficiency function that bends no more than twice. In fact, one may appeal to (5)/(9) and (13) as second-order Taylor-series expansions of any downward-sloping age-efficiency function expressed as a differential equation:

\[
E' = f(E) = f(\bar{E}) + \frac{\partial f(\bar{E})/\partial E}{\bar{E}}[E-\bar{E}] + \frac{1}{2!}\frac{\partial^2 f(\bar{E})/\partial E^2}{\bar{E}^2}[(E-\bar{E})^2 + O_3]
\]

\[= \frac{f(\bar{E})-[\partial f(\bar{E})/\partial E]\bar{E}+\frac{1}{2!}\partial^2 f(\bar{E})/\partial E^2\bar{E}^2]}{A} + \frac{[\partial^2 f(\bar{E})/\partial E^2]E}{B}[E-\frac{1}{2!}\partial^2 f(\bar{E})/\partial E^2\bar{E}^2 + O_3] + \frac{\partial f(\bar{E})/\partial E}{C}E + \frac{1}{2!}\frac{\partial^2 f(\bar{E})/\partial E^2}{C^2}E^2,
\]

where \(f(\bar{E}), [\partial f(\bar{E})/\partial E],\) and \([\partial^2 f(\bar{E})/\partial E^2]\) represent the function and its derivatives (in \(E\)) evaluated at some intermediate \(\bar{E},\) such as \(\bar{E}=\frac{1}{2}.\) The only issue is whether \([\partial f(\bar{E})/\partial E]^2>\bar{f}(\bar{E})\partial^2 f(\bar{E})/\partial E^2,\) which implies (5) or (9), or \([\partial f(\bar{E})/\partial E]^2<\bar{f}(\bar{E})\partial^2 f(\bar{E})/\partial E^2,\) which leads to (13).

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6 Given the construction-by-reflection of (10) and (16), fitting (13) by nonlinear least-squares is the same as fitting (16) by reverse nonlinear least-squares.
References


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