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CONSISTENT AGGREGATION AND CHAINING OF PRICE AND QUANTITY MEASURES

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CONSISTENT AGGREGATION AND CHAINING
OF PRICE AND QUANTITY MEASURES

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Abstract

The international movement to convert the real (deflated) components of the NIPAs to chain indexes, in order to assure timeliness, has introduced grave inconsistencies. Most importantly, the components no longer add up to the totals. The U.S. accounts which employ a Fisher chain have additional inconsistencies. For example, the product of the price and quantity indexes does not reproduce the change in nominal expenditure. The paper presents a unified approach to the construction of price and quantity measures which can be chained while maintaining every kind of consistency. The solution is based on a combination of elements from the theories of price indexes and consumer surplus.

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1. Introduction

1.1 What is the Question?

1) Measurement is at the heart of empirical science. For the natural sciences this is a truism reflected in the enormous resources devoted to the construction of nuclear accelerators, observatories, space stations and laboratories of all kinds. These exist in order to generate data based on measurement procedures derived from the relevant body of scientific theory. The data are used in turn to test and advance the theory.

2) The first half of the Twentieth Century witnessed an intensive effort to reshape economics and the social sciences in accordance with the natural science paradigm, in order to obtain a scientific basis for the solution of social problems. This effort initiated the production of the vast array of social and economic statistics which we have today, including the national income and product accounts. Many of the most prominent economists and statisticians of this era attempted to provide a theoretical rationale for these measurements. Unfortunately, these efforts failed and largely ceased, following a devastating critique by Samuelson\(^1\). As an alternative, he embraced the notion of a distributionally sensitive, ordinal social welfare function. Subsequently there developed a split between researchers at statistical agencies, concerned with the minutia of generating statistical data and academic economists producing highly abstract theories.

3) The split between theory and practice in relation to measurement is so deep and has existed for so long, that by now few economists are aware of it. One consequence is that they are unable to offer convincing arguments, based on economic theory, regarding the implications of the most important economic statistics, for example, the consumer price index and real national product.

4) The CPI became the subject of a major controversy in the United States in connection with the so called ‘Boskin Report’ (Boskin et al., 1996). The report argued that the CPI overstates the inflation rate by at least one percent. Correction of this ‘mistake’ would reduce inflation adjusted transfer payments by many billions of dollars The report had no fundamental theory on which to base the index, aside from a passing reference to a representative consumer. It equally lacked any statement of the purpose which the deflated transfer payments are supposed to serve. Grilliches (1997), himself a member of the Commission, wrote a comment titled: “What is the Question? That is the Question!”. Indeed!

5) The most common justification of neo-liberal economic policies is that they promote economic growth as measured by conventional statistics on real GDP. These assertions create the impression that the desirability of measured economic growth is somehow an implication of economic theory. This is simply false. Detailed criticisms of the preeminence given to growth have been advanced by the opponents of neo-liberalism. Fundamental economic theory cannot be found on either side of the argument.

\(^1\) As early as his dissertation, which became the *Foundation of Economic Analysis, Samuelson (1947)* radically rejected any kind of cardinal measure as meaningless and developed his revealed preference approach as an alternative. Samuelson (1942) had shown that there was no meaningful sense in which the marginal utility of income could be constant, a central assumption of traditional consumer surplus analysis. Samuelson (1950) reviewed the attempts at justifying a real national income measure on the basis of compensation measures and showed that all of them were unsatisfactory.
1.2 Theories of Price and Quantity Aggregation

6) The literature on the aggregation of price and quantity movements is vast and disparate. My focus here is on economic theories, i.e., those which assume that the data are generated by a maximizing agent. The largest and most important part of this literature deals with the individual household and attempts to measure changes in household welfare. I concentrate on this literature and omit the topic of productivity measurement at the level of the firm.

7) Apart from the economic theories, there are also axiomatic and statistical approaches to price/quantity aggregation. These have been largely replaced by the economic approach, but it is sometimes argued that they have a role to play when the economic theory is inapplicable. Given the present state of the economic theory, this could be the entire macro-level at which the empirical measure are computed. I ignore these literatures for two reasons. One is that the economic theory of this paper does apply to the macro-level, indeed this is the focus. Secondly, I believe that in an empirical situation which is ill understood, formal axioms cannot be evaluated and may be misleading. I propose instead a pragmatic approach, closely linked to the economic theory where it is applicable, but also plausible when the maximization assumption is not made.

8) There are two economic theories: the ‘economic theory of index numbers’ and consumer surplus analysis. That these two theories are unconnected is a reflection on the generally unsatisfactory state of the field. Both theories attempt to measure changes in the money metric defined in terms of the household expenditure function. Index theory attempts to do this in ratio form, CS analysis uses differences, hardly a cogent reason for having two distinct fields. I use the following terminology in this paper: index denotes a ratio, variation, a term taken from Hicks, denotes a difference, measure is used to denote either or both.

9) In their traditional formulations both of these theories are deeply flawed. The defects of index theory are:

a) The empirical measures approximate the theoretical measures only under the assumption of homothetic preferences.

b) The approximation properties do not carry over to the aggregate of consumers.

c) There is agreement on the Konüs index as the proper theoretical index of the cost-of-living, but there is no agreement on the theoretical quantity index. The favored Malmquist index is not dual to the Konüs index-deflating by one does not yield the other.

d) There is no theoretical definition of sub-totals of real magnitudes. As these are computed in practice, they do not add to the respective totals.

e) There exists no fundamental theory to deal with sequential estimation, or ‘chaining’, which is unavoidable in a time-series context. Since the recent conversion of the national income and product accounts of most nations to annually chained form, inconsistencies in these data have become a major concern of empirically oriented macro-economists. Varvares et al. (1998) mention the following inconsistencies for U.S. macroeconomic, chained time-series: The components of GDP do not sum to GDP. Real magnitudes multiplied by their deflators do not give the nominal values. Quarterly data do not add to annual data. Economic time series, as currently computed, are lacking an applicable fundamental theory and exhibit profound anomalies.

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2 Comprehensive discussions of all aspects of index theory can be found in Diewert and Nakamura (1993).
10) CS evaluations of public investment projects became hugely popular after World War II. The methodology was easy to implement and appeared to be based on economic theory. Beginning with Dupuit and continuing to the present, the ‘triangular approximation’ used in CS analysis has been interpreted as an area under a linear demand curve. An attractive feature of the formula is that it can be aggregated. The sum of household surpluses is an expression of the same form in the aggregate data. Prominent theorists have attempted analytical derivations of the triangular approximation, but without success. The present situation is a complete split between theoretical and applied economists. The former reject CS analysis as meaningless, while the latter continue to employ it.

1.3 A Theoretical Foundation for Price and Quantity Aggregation

11) This paper is part of an effort to provide a unified foundation for price/quantity aggregation, so as to either justify existing measures or improve on them. In Hillinger (2000) I argued that the only possible theoretical foundation for measuring household and aggregate welfare is in terms of two money metrics defined in terms of the household expenditure function. At the household level, the cost-of-living (CL) is defined as the expenditure required to maintain a given utility level at different prices. Real consumption (RC) is the expenditure associated with different utility levels at constant prices. The two metrics make it possible to decompose the change in nominal expenditure into a price and a quantity component. The aggregate CL (ACL) and aggregate RC (ARC) are defined by summation over the households. ACL and ARC allow a decomposition of nominal expenditure at the aggregate level.

12) A pragmatic decomposition of expenditure, which does not require any assumption about how the data were generated, is also possible. A quantity change is defined as the change in expenditure when prices are kept constant, while quantities are allowed to change. A price change is the change in the cost of a fixed basket of commodities at different prices. These are the ideas commonly employed in the construction of empirical measures.

13) The two approaches are connected by the fact that the changes computed by means of the pragmatic approach are first order differential approximations to the theoretical changes, the slopes being computed at either the beginning (Laspeyres), or end (Paasche) of the interval. A problem with these measures is that they are biased. This is a serious issue in the case of a chain measure, which would exhibit the same bias in each link of the chain. For example, a Laspeyres price index under estimates the inflation rate, because it does not take account of the intra-period substitution away from goods that have become relatively expensive. This has motivated statisticians in the United States and Canada to substitute Fisher (geometric) quantity indexes for the traditional Laspeyres quantity indexes in the construction of real national income and product statistics. While this solves the bias problem, it does not solve the other fundamental problems with such measures that were mentioned above.

14) Given the limitations of Fisher or other ‘superlative’ indexes, I was motivated to search for a solution along a different route: The basic idea was to use variations, which are additive, rather than indexes, which are not. This suggested immediately the use of centered variations, which are arithmetic averages of Laspeyres and Paasche variations, instead of Fisher indexes which are geometric averages of Laspeyres and Paasche indexes. In order to realize this research program, two fundamental issues had to be resolved:

15) All attempts at providing rigorous proofs for a quadratic approximation property of centered variations had ended in failure. In Hillinger (2000), I was able to derive several such results, as well as one on exact replication. The second issue is that of deriving a method for deflating the prices used as parameters in the quantity measures. This is a central topic for the present paper.
2. Pragmatic Aggregation of Quantities and Prices

16) The fundamental intuitive idea for the construction of a price measure is to compare the cost of a fixed basket of goods in two periods. Analogously, a measure of the average change in quantities is obtained by comparing the commodities consumed in two different periods evaluated at a common price. The comparison can take the form of a variation, or of an index. Accordingly we define:

\[QV^t = p^b \Delta x^t\]

\[PV^t = x^b \Delta p^t\]

\[QI^t = \frac{p^b x^{t+1}}{p^b x^t}\]

\[PI^t = \frac{x^b p^{t+1}}{x^b p^t}\]

17) The following definitions have been used: The superscripts ‘\(t\)’ and ‘\(t+1\)’ stand for the two periods being compared, the superscript ‘\(b\)’ stand for ‘base’; bold letters denote vectors; an expression of the form \(xp\) is an inner product.

18) Most commonly, the base vector is taken to be that of either the initial, or the final period of the comparison. In that case we speak of a Laspeyres or Paasche variation or index, respectively. These can be used to illustrate duality. Letting \(y\) devote total expenditure:

\[\Delta y^t = p^t \Delta x^t + x^{t+1} \Delta p^t = LQV^t + PPV^t\]

\[\Delta y^t = p^{t+1} \Delta x^t + x^t \Delta p^t = PQV^t + LPV^t\]

\[\frac{y^{t+1}}{y^t} = \frac{p^{t+1} x^{t+1}}{p^t x^t} \cdot \frac{x^{t+1} p^{t+1}}{x^t p^t} = LQI^t \cdot PPI^t\]

\[\frac{y^{t+1}}{y^t} = \frac{p^{t+1} x^{t+1}}{p^{t+1} x^t} \cdot \frac{x^t p^{t+1}}{x^t p^t} = PQI^t \cdot LPI^t\]

19) Duality requires that if the initial period is the base for one measure, than the final period must be the base for the other. This is a reflection of how the change in expenditure can be decomposed in reality. For example in (2.5) the decomposition involves first a move from \(p^t x^t\) to \(p^{t+1} x^{t+1}\) and then a move from \(p^{t+1} x^{t+1}\) to \(p^{t+1} x^{t+1}\).

20) Each of the two types of mathematical expression has a major advantage which the other lacks. The ratios have the advantage of being unaffected by levels and hence also of units of measurements. This makes them suitable for comparisons. For example the growth rate of a small country with an arbitrary monetary unit can be compared with that of a big country with another arbitrary monetary unit. The advantage of the variations is that they are linear in the variables and therefore additive. Addition can proceed in arbitrary stages over commodities and over agents; additivity of subtotals to the relevant aggregate is always maintained.
3. Centered Measures

21) The principal aims of this paper can be achieved by means of the Laspeyres and Paasche measures discussed in the previous sections. However, a more elegant and powerful theory can be developed using centered variations which are averages of Laspeyres and Paasche variations. Define

\[ \bar{x}' = \frac{1}{2} (x' + x'^{+1}) \quad \text{and} \quad \bar{p}' = \frac{1}{2} (p' + p'^{+1}) \]

and

(3.1) Centered Quantity Variation \[ CQV' = p' \Delta x' = \frac{1}{2} (LQV' + PQV') \]

(3.2) Centered Price Variation \[ CPV' = x' \Delta p' = \frac{1}{2} (LPV' + PPV') \].

22) These expressions are dual, so that

(3.3) \[ \Delta y' = CQV' + CPV' \].

23) The centered variations are additively consistent. They can be summed in stages over arbitrary classifications of groups of commodities and households. Since this is straightforward, I will deal in this section only with aggregation over agents. The following section leads naturally to a consideration of aggregation over commodities.

24) There are N sub-aggregates which may be collections of consumers or firms; for example, all households with certain demographic characteristics. Important in the present context is only that all sub-aggregates must face the same vectors of market prices. The i-th sub-aggregate has expenditure \( y_i \), faces market prices \( p \) and purchases the commodity vector \( x_i \). Aggregate expenditure is

(3.4) \[ y' = \sum y_i' = p' \sum x_i' = p' x' \].

25) The aggregate centered variations are defined analogously to the micro level:

(3.5) \[ ACQV' = \bar{p}' \Delta x' = \sum \bar{p}' \Delta x_i' = \sum CQV_i' \]

(3.6) \[ ACPV' = \bar{x}' \Delta p' = \sum \bar{x}' \Delta p_i' = \sum CPV_i' \].

26) Aggregation preserves duality:

(3.7) \[ \Delta y' = ACQV' + ACPV' \].

4. Theoretical Foundation

4.1 Laspeyres/Paasche Variations; Linear Approximations

27) Let \( x \) be the household consumption vector, \( p \) the corresponding price vector, \( y = px \) the household expenditure and \( u(x) \) a utility function, assumed to be twice continuously differentiable and strictly quasi-concave. The corresponding expenditure function

(4.1) \[ e(p, u) = \min_x px : u(x) \geq u \]

specifies the minimum expenditure required to reach the utility level \( u \) at prices \( p \).
28) The first money metric, referred to as real consumption, is defined as

\[ RC(u; p^b) = r(u; p^b) = e(p^b, u) \]

and measures the cost of different utility levels at a constant base period price vector.

29) The second metric is the cost-of-living, defined as

\[ CL(p; u^b) = c(p; u^b) = c(p, u^b) \]

and measures the cost of a base utility level at different prices.

30) The corresponding variations are:

\[ RCV' = r(u(x'^t), p^b) - r(u(x'), p^b) \]
\[ CLV' = c(p'^t, u(x^b)) - c(p', u(x^b)) \]

31) For suitable choices of \( p^b, x^b \), these expressions are dual:

\[ RCV'^t + CLV'^t = y'^t - y^t. \]

32) As an example, consider Laspeyres and Paasche variations:

\[ LRCV' + PCLV' = e(p^t, u(x'^t)) - e(p^t, u(x')) + e(p'^t, u(x^t)) - e(p'^t, u(x'^t)) = y'^t - y^t. \]

33) The duality relations considered thus far enable us to infer a price movement from a quantity movement and vice versa. Another set of duality relations is equally fundamental and given by the following lemmas.

34) Let \( h(p, u) \) be the Hicksian (compensated) demand function.

\[ \nabla_p e(p, u) = h(p, u) \]
\[ \nabla_x e(p, u(x)) = p. \]

35) To prove (4.9) consider the following identity which must hold in equilibrium:

\[ e(p, u(x)) = px. \]

36) Here \( p \) has two different interpretations, as a parameter of \( e(\cdot) \) and as a market price. By assumption, the two are set equal. Differentiating (4.10), keeping \( p \) constant as a parameter, yields (4.9).
37) The Lemmas of the previous section imply the following differentials of the $r(\cdot)$ and $c(\cdot)$ functions:

\begin{align*}
(4.11) & \quad LRCV' \equiv \Delta r(p') \approx \nabla_x e(p', u(x')) \Delta x' = p' \Delta x' = LQV' \\
(4.12) & \quad PRCV' \equiv \Delta r(p^{+e}) \approx \nabla_x e(p^{+e}, u(x^{+e})) \Delta x' = p' \Delta x' = PQV' \\
(4.13) & \quad LCLV' \equiv \Delta c(u') \approx \nabla_p c(p', u(x')) \Delta p' = x' \Delta p' = LPV' \\
(4.14) & \quad PCLV' \equiv \Delta c(u^{+e}) \approx \nabla_p c(p^{+e}, u(x^{+e})) \Delta p' = x^{+e} \Delta p' = P \cdot P V' .
\end{align*}

38) These variations can be aggregated by simple summation to expressions of the same form on aggregate data. They can also be converted from differences to ratios. Proceeding in this manner leads to the justification of Laspeyres and Paasche indexes as linear approximation to aggregate RC and aggregate CL. I do not pursue this straightforward analysis here. Instead, I turn to centered variations which allow a more powerful and elegant theory.

\section*{4.2 Centered Variations; Quadratic Approximation; Exact Replication}

39) In comparing two situations, the choice of either a Laspeyres or a Paasche measure appears to be arbitrary. It is therefore plausible to construct measures that are in some sense averages of the two. This is generally true of superlative indexes. For example, Fisher’s ‘ideal’ index is a geometric average of a Laspeyres and a Paasche index. The theory of this paper is based on what I refer to as centered variations, which are arithmetic averages of Laspeyres and Paasche variations. Relative to indexes, the centered variations have two fundamental advantages: They are additive across both commodities and agents; their theoretical properties do not require any assumption of homotheticity or linear homogeneity.

40) The decisive advantages connected with centered variations come with a certain cost in the complexity of deriving the basic theoretical results. These fall under two headings: approximation, which is the subject of the present section and deflation, which is treated in the next. Fortunately, this complexity does not extend to the derived formulas and their empirical implementation.

41) The problem of relating centered empirical variations to theoretical variations defined in terms of the household expenditure function has a long history in consumer surplus analysis. However, the first complete and satisfactory solution was given in Hillinger(2000). It turns out that corresponding to the single pair of empirical centered variations, there exists an infinity of centered pairs of theoretical variations, all of which can be defined in a natural manner. I limit myself here to stating two sets of results.

42) The following two equations refer to quadratic approximations:

\begin{align*}
(4.15) & \quad CRCV'\left(\bar{p}'\right) = r\left(\bar{p}', x^{+e}\right) - r\left(\bar{p}', x'\right) \overset{QA}{\approx} \bar{p}' \Delta x' = CQV' \\
& \quad CCLV'\left(\bar{u}'\right) = c\left(\bar{u}', p^{+e}\right) - c\left(\bar{u}', p'\right) \overset{QA}{\approx} \bar{x}' \Delta p' = CPV',
\end{align*}

with $\bar{u}' = u\left(x\left(\bar{p}', \bar{y}'\right)\right)$.
43) The following result is on exact replication:

\[ \exists \mathbf{p}^* \in \left( \mathbf{p}', \mathbf{p}'^* \right), \ u^* \in \left( u', u'^* \right), \text{ such that}. \]
\[ \text{CRCV}^* \left( \mathbf{p}^* \right)^{\text{ER}} = \text{CQV}^* \]
\[ \text{CCLV}^* \left( u^* \right)^{\text{ER}} = \text{CPV}^*. \]

44) For simplicity, I will in the remainder of this paper make use only of equations (4.16). Note that the
duality of the empirical variations CQV' and CPV' assures the duality of the theoretical variations
CRCV' and CCLV'.

5. Aggregating Over Households

45) The purpose of this section is to show that the results obtained for the individual household hold in the
same form for an aggregate of households. This is a principal advantage of working with variations,
rather than indexes. Assume that there are N households. The i-th household has expenditure \( y_i \),
faced market prices \( \mathbf{p} \) and purchases the commodity vector \( \mathbf{x}_i \). Aggregate expenditure i

\[ (5.1) \quad y = \sum y_i = \mathbf{p} \sum x_i = \mathbf{px}. \]

46) The aggregate centered variations are defined analogously to the micro level:

\[ (5.2) \quad \text{ACQV} = \overline{\Delta} = \sum \overline{\mathbf{p}} \Delta \mathbf{x} = \sum \text{CQV}_i \]
\[ (5.3) \quad \text{ACPV} = \overline{\mathbf{x}} \Delta \mathbf{p} = \sum \overline{\mathbf{x}}_i \Delta \mathbf{p} = \sum \text{CPV}_i \]

47) Addition also preserves duality:

\[ (5.4) \quad \Delta y = \text{ACQV} + \text{ACPV} \]

48) Aggregate RC and CL variations are also defined by summation over the households:

\[ (5.5) \quad \text{ARCV} \left( \mathbf{p}^* \right) = \sum \text{RCV}_i \left( \mathbf{p}^* \right) \]
\[ (5.6) \quad \text{ACLV} \left( \mathbf{u}^* \right) = \sum \text{CLV}_i \left( \mathbf{u}^* \right), \quad \mathbf{u}^* = \left( u_1, ..., u_N \right). \]

49) The interpretation of these aggregate measures is as follows: \( \text{ARCV} \left( \mathbf{p}^* \right) \) is the cost, at \( \mathbf{p}^* \) prices,
of moving each household from its initial to its final utility level. \( \text{ACLV} \left( \mathbf{u}^* \right) \) is the cost of
maintaining each households utility at a level \( u_i^* \) for a given price change.

50) Combining (5.3-4) and (5.5-6):

\[ (5.7) \quad \text{ARCV} \left( \mathbf{p}^* \right) = \text{ACQV} \]
\[ (5.8) \quad \text{ACLV} \left( \mathbf{u}^* \right) = \text{ACLV}. \]

51) It is remarkable that the aggregate theoretical variations are measured exactly by the empirical
variations defined on aggregate data.
6. Chaining and Deflating

52) In the previous section I discussed replication of the theoretical measures, but I deferred discussing an issue which involves meaningfulness of these measures. Consider \( RCV'(p^*) \) and \( CQV'(\bar{p}') \). These expressions are linear homogeneous in the base price vector. Nominal variations of prices and money incomes, which do not effect quantities, could make these expressions arbitrarily large or small. This is not itself a problem, rather it is indicative of the fact that the price vectors supply the units of measurement. The situation is, however, more complicated than is the case with single dimensional, invariant scales, such as physical scales of length or weight. The price vector necessarily changes from period to period and we need to take account of the changes in relative prices. At the same time we need to deflate in order to remove changes in the price level, which have the effect stretching or squeezing the scale. The problem is particularly acute in a time-series context, in which successive quantity variations are added. Even in a single comparison of two situations, it makes little sense to compute \( \bar{p}' \) or \( p^* \) if \( p^t \) and \( p^{t+1} \) have substantially different price levels.

53) It has turned out that to compute a quantity measure we must first compute a price measure. Fortunately, the price measure can be computed directly. Consider \( CLV'(x^*') \) and \( CQV'(\bar{x}') \). Here the commodity weights are real and invariant to any purely nominal changes.

54) The deflator should be chosen in accordance with the theory developed thus far. This requirement is in fact sufficient to determine the deflator. Since a deflator, by definition, measures the inflation between two points, after deflation measured inflation must be zero. Letting \( \lambda' \) be the deflator, which measures the price increase from period \( t-1 \) to period \( t \). The formal requirement is

\[
\text{(6.1)} \quad \bar{x}^{t-1}(p'/\lambda' - p^{t-1}) = 0
\]

which implies that

\[
\text{(6.2)} \quad \lambda' = \frac{\bar{x}^{t-1}p'}{\bar{x}^{t-1}p}.
\]

55) The definition of the deflator does not as yet solve the problem of deflating in a time-series context. Here we need to deflate the prices of each period back to the base period level. To accomplish this define \( \lambda^{0,1} \) as the deflator to be used for deflating \( p^t \) back to the level of \( p^0 \). It is defined by

\[
\text{(6.3)} \quad \lambda^{0,1} = \lambda^1, \quad \lambda^{0,j} = \lambda^{0,j-1} \cdot \lambda',
\]

which implies

\[
\text{(6.4)} \quad \lambda^{0,j} = \lambda^j \cdots \lambda'.
\]

56) Having derived the deflator, we could turn again to the quantity variations in order to compute changes in real expenditures explicitly, using deflated prices as weights. This will be done below, but it is useful to look first at the implicit definition of real expenditures as corresponding to deflated nominal expenditures. Define

\[
\text{(6.5)} \quad \bar{p}' = p'/\lambda^{0,j}, \quad \bar{y}' = y'/\lambda^{0,j}.
\]
Let $I$ denote a subset of commodities and define

$$\bar{p}^y = \frac{p^y}{\lambda^{0,y}}, \quad \bar{y}^y = \frac{y^y}{\lambda^{0,y}}.$$  

We can also write

$$\bar{y}^y = \frac{\bar{p}^y x', \quad \bar{y}^y = \frac{\bar{p}^y x'}{\lambda^{0,y}}.$$  

The deflated magnitudes measure the aggregate and any sub-aggregate, which can be a single commodity, in real terms. That these magnitudes satisfy the following

(6.8) **Fundamental Postulate for Real Magnitudes**: Let $q_k$ designate a real magnitude, with $i_k \in I_k$ and $I_k \subset I$. and let $q_k$ be the expenditure of the set of commodities in $I_k$. The real magnitudes must satisfy the following conditions:

a) The ratio $\frac{q_i}{q_k}$ must reflect the market exchange rate between these commodity bundles.

b) The expenditure $q_k$ must reflect the price level of a designated base period.

c) Additivity must be maintained, in the sense that $q_k = \sum q_k$ and $q_k = \sum q_i$.

It is clear that the deflated expenditures in (6.7) satisfy the postulate. Conversely, only a uniform deflator applied to all prices will produce expenditures satisfying the postulate. The postulate is so elementary as to seem trivial, but no one appears to have thought of it. The procedure universally followed by statisticians is to compute the sub-aggregates by the explicit method, using the same quantity index formula employed in computing the aggregate. A consequence is that the sub-totals do not add up to the grand total. What has been less obvious is that the approach is fundamentally flawed. When sub-totals are computed independently, no account can be taken on inter-group price ratios. The fundamental postulate can therefore not be satisfied.

While the explicit method does not work with indexes it does work with variations using deflated prices. To compute them in this manner had been my original motivation for devising the deflator. To go via this second route brings additional benefits:

a) The two approaches are shown to be consistent.

b) An interesting decomposition of the total changes in real expenditures is obtained.

c) The empirical measures can be linked more explicitly to their theoretical counterparts.

I discuss first the variations at the aggregate, then at the sectoral level. The variations in (6.9) are based on a decomposition of the total change in expenditures into two parts: $\bar{y}^y - y^{y-1}$ is the aggregate real expenditure variations. $y^y - \bar{y}^y$ is the aggregate price variation i.e. the purely nominal expenditure increase due to changes of all prices in the proportion $\lambda'$. The third line shows a decomposition of AREV into an aggregate quantity variation and a relative price variation. The RPV at the aggregate level is zero, by construction of the deflator. Therefore, AREV=AVQ. This differs at the sectoral level. For APV, quantities remain fixed at $x'$ and the expression implies $y' / \bar{y}' = \lambda'$.

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3 For a discussion see Hill(1993).
(6.9) \( y^{t+1} - y^t = \tilde{y}^{t+1} - \tilde{y}^t + y^{t+1} - \tilde{y}^{t+1} \)

\[ = \text{AREV}^t + \text{APV}^t \]

\[ = \frac{1}{2} \left( \tilde{p}_i^{t+1} + p_i^t \right) \left( x_i^{t+1} - x_i^t \right) + \tilde{x}_i^{t+1} \left( p_i^{t+1} - \tilde{p}_i^t \right) + x_i^t \left( p_i^t - \tilde{p}_i^t \right) \]

\[ = A\text{QV}^t + \text{RPV}^t + \text{APV}^t \]

63) Using (5.7-8), we can link the empirical to the theoretical measures:

(6.10) \( \text{ARCV} \left( p^{g'}, x^{r+1} - x^t \right) = A\text{QV}^t, \quad p^{g'} \in \left( p^t, \tilde{p}^{t+1} \right) \)

\( \text{ACLV} \left( u^{g'}, \tilde{p}^{t+1} - p^t \right) = \text{RPV}^t, \quad u^{g'} \in \left( u^t, u^{t+1} \right) \)

\( \text{ACLV} \left( u^t; p^t - \tilde{p}^t \right) = \text{APV}^t \)

64) Next I turn to the question how this method of deflation works out when there are sub-aggregates. A particular sector will be denoted by the subscript \( I \). Here there is a novelty in that the sectoral relative price change will generally differ from zero. This is because the deflator was designed to make the comparable aggregate expression equal to zero, not any sectoral measure. This makes it desirable to introduce a new terminology. The change \( \tilde{y}^t_i - y^t_i \) is now referred to as the sectors real expenditure variation. It is decomposed into a quantity variation and a relative price variation. The interpretation is that \( \text{REV}^t_i \) would increase with a quantity increase at constant relative prices, but also with an increase of the sectors relative price with unchanged quantities.

(6.11) \( y^t_i - y^{t-1}_i = \tilde{y}^t_i - y^{t-1}_i + y^t_i - \tilde{y}^t_i \)

\[ = \text{REV}^t_i + \text{APV}^t_i \]

\[ = \frac{1}{2} \left( \tilde{p}_i^t + p_i^{t-1} \right) \left( x_i^t - x_i^{t-1} \right) + \tilde{x}_i^{t-1} \left( \tilde{p}_i^t - p_i^{t-1} \right) + x_i^t \left( p_i^t - \tilde{p}_i^t \right) \]

\[ = Q\text{V}^t_i + \text{RPV}^t_i + \text{APV}^t_i \]

65) A theoretical equivalence, analogous to (6.10), can be given for the sectoral variations also. They should be interpreted as partial differentials.

66) It would be interesting to explore the empirical implementation of the decomposition of \( \text{REV}^t_i \), but I will not pursue this topic further here.\(^4\)

7. Conclusion

67) The paper demonstrates that a powerful unified theory of price and quantity measures is possible. The theory is free of anomalies that afflict current measures, it is applicable at all levels of aggregation and it is connected to fundamental microeconomic theory.

\(^4\) A problem with the direct implementation of (6.10) is that it could lead to the computation of negative quantities if both prices and quantities fall sharply.
References


(1947), Foundation of Economic Analysis, Cambridge, Harvard University Press.
