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Adjustment of Monthly or Quarterly Series to Annual Totals: An Approach Based on Quadratic Minimization

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This article considers the problem of adjusting monthly or quarterly time series to make them accord with independent annual totals or averages without introducing artificial discontinuities. A general approach and some specific procedures involving constrained minimization of a quadratic form in the differences between revised and unrevised series are proposed. Some computational advantages are noted. Attention is given to the relationships between the adjustment problem and earlier work by other authors on the creation of monthly or quarterly series when only annual figures are available. An example is provided to illustrate the application of the proposed adjustment procedures.

1. INTRODUCTION

A problem often encountered in the preparation of economic time series is that of adjusting monthly or quarterly values obtained from one source to make them accord with annual totals or averages obtained from another. For example, a frequently conducted survey restricted to larger business establishments might be the source of information on short-term (intra-annual) fluctuations of employment, output, profits, etc., while a survey covering all business establishments might provide annual figures. Typically, one would wish to make the adjustment in such a way as to distort as little as possible the original month-to-month or quarter-to-quarter movements of the series. A simple procedure is to distribute the discrepancy for a given year among the periods within that year, either in equal amounts or *pro rata*. However, if the discrepancies are not uniform from year to year, this procedure will introduce an artificial step or discontinuity between the last period of one year and the first period of the next. The purpose of the present article is to pose the adjustment problem as one of constrained quadratic minimization and to suggest some specific procedures which avoid the difficulty just noted.[1]

The problem of adjusting a given monthly or quarterly series to accord with a set of annual figures is related to another problem, that of *creating* a monthly or quarterly series, given *only* a set of annual figures. The latter problem has been explored by Lisman and Sandee [6], Glejser [5], and Boot, Feibes and Lisman [2]. In particular, Boot, Feibes and Lisman have suggested an approach which, in essence, may be viewed as a special application of the general approach to the adjustment problem that is proposed here. This matter is taken up in Section 4.

2. FORMULATION OF THE PROBLEM AND A GENERAL APPROACH TO ITS SOLUTION

Assume that we are concerned with intra-annual time periods of which there are k per year, k being an integer. Let the time series of interest cover m years and consist of $n=mk$ values. The original values are represented in column-vector form by $z=[z_1\ z_2 \cdots z_n]'$. Assume also that we have, from a different source, a set of m annual totals[2] represented by $y=[y_1\ y_2 \cdots y_m]'$. The problem is to adjust the original vector z to obtain a new vector $x=[x_1\ x_2 \cdots x_n]'$ by a method which (a) minimizes the distortion of the original series, in some sense, and (b) satisfies the condition that the k values of the new series within each year sum to the given annual total for that year. More formally, we specify a penalty function, $p(x, z)$, and express the problem as that of choosing x so as to minimize $p(x, z)$ subject to

$$\sum\nolimits_{(T-1)k+1}^{Tk} x_t = y_T \qquad \text{for } T = 1, 2, \cdots, m.$$

Consider the class of penalty functions represented by $(x-z)'A(x-z)$, a quadratic form in the differences between the original and adjusted time-series values, A being a symmetric $n \times n$ nonsingular matrix to be specified later. We set up a Lagrangian expression and write

$$u = (x - z)'A(x - z) - 2\lambda'(y - B'x) \qquad (2.1)$$

where

$$\lambda = [\lambda_1\lambda_2 \cdots \lambda_m]' \quad \text{and} \quad B = \begin{bmatrix} j & o & \cdots & o \\ o & j & \cdots & o \\ \vdots & \vdots & & \vdots \\ o & o & \cdots & j \end{bmatrix},$$

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[1] The problem is widely recognized in government statistical agencies, and it appears that a variety of specific procedures have been devised by government statisticians and others for coping with it. One approach of long standing was proposed by Bassie [1]. Friedman [4] has dealt extensively with a related but different problem, that of interpolating time series by employing other series which are correlated with them. Friedman does not deal in any detail with the problem of *allocating* annual totals or averages among months or quarters, which is the problem to which the present article is addressed.

[2] It will be assumed henceforth that the annual figures are in the form of *totals*. This means simply that if the original annual figures were averages, they will have been converted to totals by multiplying by k.

j being a k-dimensional column vector in which each element is unity and o being a k-dimensional null column vector. B is $n \times m$. The penalty-minimizing solution is obtained by taking partial derivatives of u with respect to the elements of x and λ, equating them to zero, and solving. For convenience, we write $r = y - B'z$ for the vector of discrepancies between the two sets of annual totals and express the solution in the form

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A & B \\ B' & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ B' & I \end{bmatrix} \begin{bmatrix} z \\ r \end{bmatrix}, \tag{2.2}$$

where I is the $m \times m$ identity matrix and 0 is the $m \times m$ null matrix. (It is assumed, of course, that the second-order conditions necessary for the solution to be a minimum are satisfied.) Using a well-known result for deriving the inverse of a partitioned matrix, the solution for x is then found to be $x = z + Cr$, where $C = A^{-1}B(B'A^{-1}B)^{-1}$. Thus the adjusted values are equal to the original values plus linear combinations of the discrepancies between the two sets of annual totals.

3. SOME SPECIFIC SOLUTIONS

Suppose, first, that A is the identity matrix, which means that we are minimizing the sum of squares of the simple differences between the original and revised values. In this case, $C = 1/k\, B$, implying that the penalty function is minimized by distributing the discrepancy for each year in equal amounts among the k periods within the year. As noted before, this results, in general, in a spurious step or discontinuity between the last period of one year and the first period of the next. Clearly, $A = I$ is likely to be a bad specification.

A more appealing possibility is to employ a penalty function based on the differences between the *first differences* of the original and adjusted series:

$$p(x, z) = \sum_{t=1}^{n} (\Delta x_t - \Delta z_t)^2 = \sum_{t=1}^{n} [\Delta(x_t - z_t)]^2. \tag{3.1}$$

If Δ is defined as a backward difference operator (e.g., $\Delta x_t = x_t - x_{t-1}$), the penalty function will include the term $\Delta(x_1 - z_1) = (x_1 - z_1) - (x_o - z_o)$, the subscript o referring to the last period of the year preceding year 1. Period o is outside the range over which the series is to be adjusted and we may take x_o and z_o as equal and write $\Delta(x_1 - z_1) = x_1 - z_1$. (It is assumed that no adjustments are to be made to the original series for years outside the range from year 1 to year m, inclusive.) The vector of (backward) first differences may then be expressed as $D(x - z)$, where D is an $n \times n$ matrix given by

$$D = \begin{bmatrix} 1 & 0 & 0 \cdots & 0 & 0 \\ -1 & 1 & 0 \cdots & 0 & 0 \\ 0 & -1 & 1 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & -1 & 1 \end{bmatrix}. \tag{3.2}$$

The quadratic form to be minimized, subject to the annual constraints, is now $(x - z)'D'D(x - z)$. In short, $A = D'D$.

It is also possible to specify the penalty function in terms of the differences between the second or higher-order differences of the original and adjusted series. Specifying the function in general form,

$$\begin{aligned} p(x, z) &= \sum_{t=1}^{n} (\Delta^h x_t - \Delta^h z_t)^2 \\ &= \sum_{t=1}^{n} [\Delta^h (x_t - z_t)]^2, \end{aligned} \tag{3.3}$$

where Δ^h is the hth-difference operator. Eliminating from the penalty function all values outside the adjustment range by setting $x_t = z_t$ for $t = 0, -1, \cdots, 1-h$, the vector of hth differences is found by h successive applications of the D matrix: $DD \cdots D(x - z)$, so that $A = D'D' \cdots D'D \cdots DD$.

One of the attractive features of the foregoing approach is that the matrices have some convenient properties which make them easy to work with. Let R be an upper triangular matrix in which every element on or above the principal diagonal is 1 and every element below the principal diagonal is zero. Using the result $D^{-1} = R'$, which may be verified by inspection, it is easily shown that $(D'D)^{-1} = R'R$, $(D'D'DD)^{-1} = R'(D'D)^{-1}R$, etc., each inverse in the series being obtained from the previous one by premultiplying by R' and post-multiplying by R. This makes it unnecessary to invert A, which is an $n \times n$ matrix and may be very large. For example, if $A = D'D$, we can write the inverse immediately:

$$A^{-1} = (D'D)^{-1} = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & 2 & 2 \cdots & 2 \\ 1 & 2 & 3 \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 \cdots & n \end{bmatrix}. \tag{3.4}$$

Of course, the calculations still require the inversion of the $m \times m$ matrix $B'A^{-1}B$. However, this is a much less formidable task than inverting an $n \times n$ matrix, as one would have to do if the A matrix were less well behaved. For example, if one were working with 25 years of monthly data, the A matrix would be 300×300 whereas $B'A^{-1}B$ would be only 25×25.[3]

4. THE PROBLEM OF CREATING A SERIES AS A SPECIAL CASE OF THE ADJUSTMENT PROBLEM

As noted already, several authors have dealt with the problem of generating a monthly or quarterly series, given only a set of annual totals. In particular, Boot, Feibes, and Lisman [2] have suggested that a reasonable procedure is to choose values for the series such that the sum of squares of either the first or second differences is a minimum, subject to the annual constraints. To see the relationship of this procedure to the adjustment procedure described earlier, first let the original series to be adjusted have constant values: $z_t = z_{t-1}$ for all relevant

[3] For further discussion of computational simplifications, see [3].

ILLUSTRATIVE EXAMPLES INVOLVING ADJUSTMENT OF ARTIFICIAL SERIES OF 5 YEARS' DURATION BASED ON ALTERNATIVE SPECIFICATIONS OF QUADRATIC PENALTY FUNCTION AND USING BACKWARD DIFFERENCES

Year and quarter	Original series	Adjusted series based on penalty function involving -							
		$(x-z)$	$\Delta(x-z)$	$\Delta^2(x-z)$	$\Delta^3(x-z)$	$(\frac{x-z}{z})$	$\Delta(\frac{x-z}{z})$	$\Delta^2(\frac{x-z}{z})$	$\Delta^3(\frac{x-z}{z})$
Year 1 - 1Q	50	75	67	62	59	56	57	55	54
2Q	100	125	127	125	123	122	124	122	120
3Q	150	175	180	182	184	200	194	194	195
4Q	100	125	126	130	134	122	125	129	132
Year 2 - 1Q	50	50	65	70	74	50	58	61	62
2Q	100	100	105	106	107	100	107	109	111
3Q	150	150	145	142	141	150	146	145	144
4Q	100	100	85	81	78	100	89	85	83
Year 3 - 1Q	50	25	27	24	22	44	40	39	38
2Q	100	75	73	72	71	78	74	73	72
3Q	150	125	123	124	125	100	109	110	111
4Q	100	75	78	80	81	78	77	78	79
Year 4 - 1Q	50	50	37	38	39	50	43	43	43
2Q	100	100	96	96	96	100	94	94	94
3Q	150	150	154	155	154	150	153	154	153
4Q	100	100	112	112	111	100	110	110	110
Year 5 - 1Q	50	75	69	68	67	56	58	58	58
2Q	100	125	124	123	123	122	123	121	122
3Q	150	175	178	178	178	200	190	189	190
4Q	100	125	129	132	132	122	129	131	130

values of t. Then $\Sigma[\Delta(x_t-z_t)]^2=\Sigma(\Delta x_t)^2$. Again, if the original series has constant first differences ($\Delta z_t=\Delta z_{t-1}$), then $\Sigma[\Delta^2(x_t-z_t)]^2=\Sigma(\Delta^2 x_t)^2$. Thus the first-difference and second-difference versions of the penalty function already proposed for *adjusting* a series reduce to the penalty functions proposed by Boot, Feibes, and Lisman for *creating* a series in cases where there exist no initial estimates of monthly or quarterly movements other than a possible estimate of linear trend.

5. ALTERNATIVE SPECIFICATIONS OF THE PENALTY FUNCTION

The penalty function for the adjustment problem can be specified in terms of proportionate differences between the original and revised series instead of arithmetic differences. (This might be preferred sometimes on the grounds that larger values in the original series should bear a commensurately larger share of the adjustment burden.) We define the proportionate difference in period t as $(x_t-z_t)/z_t$. Letting Z be the $n\times n$ diagonal matrix with diagonal elements $z_1, z_2, \cdots, z_n$, the new penalty function can be written in the form $(x-z)'Z^{-1}AZ^{-1}(x-z)$. We now minimize

$$u=(x-z)'Z^{-1}AZ^{-1}(x-z)-2\lambda'(y-B'x) \quad (5.1)$$

with respect to the elements of x and λ and, following the same route as in Section 2, obtain the result $x=z+ZA^{-1}ZB(B'ZA^{-1}ZB)^{-1}r$. It may be noted that whereas before the adjustment coefficients were independent of the values of the particular series being adjusted, this is no longer the case.

Another type of respecification involves defining the difference operator Δ (more generally, Δ^h) as a forward operator (e.g., $\Delta x_t=x_{t+1}-x_t$). The foregoing development has been based on the use of a backward difference operator, but can easily be adapted to the case of a forward operator.[4]

6. AN ILLUSTRATIVE EXAMPLE

By way of illustration, the table displays the results of adjusting a five-year artificial quarterly series to a set of artificial annual totals, using backward differences and various specifications of the penalty function. The artifi-

[4] A case can also be made for averaging the adjustment coefficients based on backward and forward differences in some circumstances. Discussion of the use of backward, forward, or average coefficients is contained in [3].

cial series chosen for this purpose has a regular and pronounced seasonal pattern but otherwise is invariant from year to year. It has values 50, 100, 150, and 100 in the four calendar quarters, so that the series sums to 400 in each year. The annual totals to which the series is to be adjusted are assumed to be 500, 400, 300, 400, and 500 in the five successive years. Additional illustrative material is provided in [3].

REFERENCES

[1] Bassie, V. Lewis, "Interpolation Formulae for the Adjustment of Index Numbers," Paper presented at the *Annual Meetings of the American Statistical Association*, December 1939.

[2] Boot, J. C. G., Feibes, W. and Lisman, J. H. C., "Further Methods of Derivation of Quarterly Figures from Annual Data," *Applied Statistics*, 16, No. 1 (1967), 65–75.

[3] Denton, Frank T., "Quadratic Minimization Procedures for Adjusting Monthly or Quarterly Time Series to Independent Annual Totals," McMaster University Department of Economics Working Paper 70-01, January 1970 (unpublished).

[4] Friedman, Milton, "The Interpolation of Time Series by Related Series," *Journal of the American Statistical Association*, 57 (December 1962), 729–57.

[5] Glejser, H., "Une Méthode d'Évaluation de Données Mensuelles à Partir d'Indices Trimestriels ou Annuels," *Cahiers Économiques de Bruxelles*, No. 19 (1966), 1e Trimestre, 45–54.

[6] Lisman, J. H. C. and Sandee, J., "Derivation of Quarterly Figures from Annual Data," *Applied Statistics*, 13, No. 2 (1964), 87–90.