Pension Funds, Life-Cycle Asset Allocation and Performance Evaluation *

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March 2009

Abstract

We present a life-cycle model for pension funds' optimal asset allocation, where the agents' labor income process is calibrated to capture a realistic hump-shaped pattern and the available financial assets include one riskless and two risky assets, with returns potentially correlated with labor income shocks. The sensitivity of the optimal allocation to the degree of investors' risk aversion and the level of the replacement ratio is explored. Also, the welfare costs associated with the adoption of simple sub-optimal strategies ("age rule" and "$1/N$ rule") are computed, and new welfare-based metrics for pension fund evaluation are disussed.

*This paper is part of a research project on "*Optimal Asset Allocation For Defined Contribution Mandatory Pension Funds*" sponsored by the World Bank and OECD. All the simulations have been carried out using a MATLAB code written by C. Fugazza. We thank, without implicating, Roy Amlan, Pablo Antolin, Zvi Bodie, Frank De Jong, Rudolph Heinz, Theo Nijman, and other participants at the OECD-WB Conference on "Performance of Privately Managed Pension Funds" (Mexico City, January 2009). An anonymous referee provided many useful comments on a previous draft of the paper.

1 Introduction

Methods for evaluating the performance of defined contribution (DC) pension funds are similar to those applied to mutual funds, and typically associate a higher return per unit of risk with better performance. These methods are adequate if a worker, or the pension fund acting on her behalf, has preferences defined exclusively over the mean and the variance of portfolio returns[1]. Ideally, though, a worker contributes to a pension fund in order to help stabilize consumption during retirement years, given that the yearly pension transfer granted by typical first-pillar schemes is lower than the last wage. Thus the optimal asset allocation ought to take into account, together with the asset return distributions and the risk aversion parameter that enter a standard portfolio choice problem, both any pension transfer accruing after retirement as well as the worker's life expectancy. Since the pension transfer is usually a fraction of labor income earned during the last working year (which is, in turn, the outcome of the worker's risky professional history) the optimal asset allocation trades off the gains from investing in high risk premium assets with the needs to hedge labor income shocks.

Adopting an explicit life-cycle perspective, this paper presents a simple model that is calibrated to deliver quantitative predictions on optimal portfolio allocation for DC pension funds. It then proposes a new, welfare-based metric in order to evaluate their performance. Our model belongs to the literature on strategic asset allocation for long-term investors. The recent expansion of defined-contribution pension schemes, with respect to defined-benefit plans, and the ensuing focus on optimal investment policies, is one of the motivations behind its growing importance. In this research area modern finance theory, as summarized for example by Campbell and Viceira (2002), has made substantial progress over the traditional (mean-variance, one-period) approach that still forms the basis for much practical financial advice. Long investment horizons, the presence of risky labor income, and of illiquid assets such as real estate, have been gradually incorporated into the analysis of optimal portfolio choice. Moreover, the conditions under which conventional financial advice (such as the suggestion that investors should switch from stock into bonds as they age, and that more risk-averse investors should hold a larger fraction of their risky portfolio in bonds than less risk-averse investors) is broadly consistent with optimal asset allocation policies have been clarified. The key intuition is that optimal portfolios for long-term investors may not be the same as for short-term investors, because of a dif-

[1] The investor may also have more elaborate preferences that, combined with investment opportunities, reduce to mean variance preferences.

ferent judgement of assets' riskiness, and because of the crucial role played by (nontradable) human wealth in the investors' overall asset portfolio.

In more detail, our life-cycle model features two risky and one riskless assets, which are parameterized by the first two moments of their return distribution, and correspond in our simulations to domestic stocks, bonds and bills. As in Bodie, Merton and Samuelson (1992) and Cocco, Gomes and Maenhout (2005), early in the worker's life the average asset allocation is tilted towards the high risk premium asset, because labor income provides an effective hedge against financial risks. On the contrary, in the two decades before retirement, it gradually shifts to less risky bonds, because income profiles peak at around age 45.

Although these patterns are associated to given values of the parameters that describe both workers' human capital and investment opportunities, as well as the institutional framework, we perform sensitivity analysis along several important dimensions. The first examines the reaction of optimal asset allocation to the labor income profile. For instance, a construction worker may face a higher variance of uninsurable labor income shocks than a teacher (Campbell, Cocco, Gomes and Menhout 2001); alternatively, the correlation between stock returns and labor income may be higher for a self-employed or a manager than for a public sector employee. If such differences have negligible effects on optimal asset allocation, the pension plan may offer the same option to all partcipants. Instead, in our simulations optimal portfolio shares are highly heterogeneous across coeval agents (despite their common life expectancy, retirement age and replacement ratios) due to such individual-specific labor income shocks. Dispersion decreases as workers approach retirement, the more so the higher is the labor income-stock return correlation: as this increases, the histories of labor incomes tend to converge over time and so do the optimal associated portfolio choices. These results suggest that the optimal allocation ought to be implemented through diversified investment options for most occupations and age brackets.

The pension transfer in our model is a fixed annuity (granted by an unmodelled first pillar or defined-benefit scheme)[2] and proportional to labor income in the last working year. Replacement ratios vary widely across countries, as documented by OECD (2007), ranging from 34.4% in UK to 95.7% in Greece. Such differences also depend on the inflation coverage of pension annuities, which is often imperfect, implying a reduced average replacement ratio. By measuring the sensitivity of optimal portfolio composition with respect to the replacement ratio, we understand whether optimal pension

[2] Koijen, Nijman and Werker (2006) argue that it is suboptimal relative to alternative annuity designs, despite its diffusion across pension systems.

fund portfolio policies should vary across countries for given members' types. When the replacement ratio falls, simulations reveal that agents save more during their working life in anticipation of lower pension incomes, thus accumulating a higher level of financial wealth. This determines a lower optimal share of stocks at all ages and for all values of the labor income-stock return correlation, holding risk aversion unchanged: with higher financial wealth, a given labor income becomes less apt to offset bad financial outcomes. In other words, our model indicates that asset allocation in low replacement ratio countries ought to be more conservative because workers' contributions to pension funds ought to be higher.

Computing the optimal life-cycle asset allocation allows to use it as a performance evaluation benchmark, which explicitly accounts for pension plan role in smooting participants' consumption risk. We propose several indicators to evaluate pension funds' performance. The first metric takes the ratio of the worker's ex-ante maximum welfare under optimal asset allocation to her welfare under the pension fund actual asset allocation: the higher the ratio, the worse the pension fund performance. Importantly, bad performance may derive not only from a lower return per unit of financial risk earned by the pension fund manager - which is what previous methods look at - but also from a bad matching between the pension fund portfolio and its members' labor income and pension risks.

Unmodelled costs of tailoring portfolios to age, labor income risk and other worker-specific characteristics can be quite high for pension funds. This is why we assess the welfare costs of implementing two simpler strategies, namely an "age rule" and a strstegy with portfolio shares fixed at 1/3 for each of our three financial assets, echoing the "$1/N$ rule" of DeMiguel, Garlappi and Uppal (2008), that outperforms several portfolio strategies in *ex post* portfolio experiments. The latter strategy performs consistently better than the "age rule", making it a better benchmark for evaluating the performance of pension funds. Importantly, our numerical results suggest that this portfolio strategy is likely to be cost-efficient for both high wealth and highly-risk-averse-average-wealth workers in medium-to-high replacement ratios countries. In these cases, the welfare costs of the suboptimal 1/3 rule are often lower than 50 basis points per annum in terms of welfare-equivalent consumption, which is likely to be lower than the management cost differential. Thus, 1/3 may well become the benchmark asset allocation in the welfare metric for performance evaluation.

The present contribution is organized as follows. The main theoretical principles that may be relevant for pension funds strategic asset allocation are outlined in Section 2. Section 3 presents our simple operative life-cycle model, showing how it can be calibrated to deliver quantitative predictions on

optimal portfolio allocation. The welfare metrics for pension funds' performance evaluation are introduced and discussed in Section 4. A final section summarizes the main conclusions.

2 The effects of the investment horizon and labor income on portfolio choice

Basic financial theory provides simple asset allocation rules for an investor maximizing utility defined over expected (financial) wealth at the end of a single-period horizon ($E_t W_{t+1}$) and no labor income, under specific assumptions on the form of the utility function and on the distribution of asset returns. In particular, when a *constant* degree of *relative risk aversion* is assumed (a simplifying assumption broadly consistent with some long-run features of the economy, such as the stationary behaviour of interest rates and risk premia in the face of long-run growth in consumption and wealth), i.e. investors have power utility, and returns are lognormally distributed, the investor trades off mean against variance in portfolio returns, obtaining (in the case of one risky asset) the following optimal portfolio share:

$$\alpha_t = \frac{E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2}}{\gamma\, \sigma_t^2}$$

where $r_{t+1} = \log(1 + R_{t+1})$ and $r^f_{t+1} = \log(1 + R^f_{t+1})$ are the continuously compounded returns on the risky and riskless asset respectively, σ_t^2 is the conditional variance of the risky return, and γ is the constant relative risk aversion parameter[3]. This result is equivalent to the prediction of the simple mean-variance analysis, and the equivalence extends also to the case of many risky assets, with γ affecting only the scale of the risky asset portfolio but not its composition among different asset classes.

The optimal investment strategy may substantially differ from the above one-period, "myopic", rule if the investment horizon extends over multiple periods and when a human wealth component is added to financial wealth. We briefly consider those two cases in turn.

[3] When $\gamma = 1$ the investor has log utility and chooses the portfolio with the highest log return; when $\gamma > 1$ the investor prefers a safer portfolio by penalizing the return variance; when $\gamma < 1$ the investor prefers a riskier portfolio.

2.1 Multi-period investment horizons

When the investor has a long-term investment horizon, maximizing the expected utility of wealth K periods in the future ($E_t W_{t+K}$), returns are log-normally distributed, and the investor is allowed to rebalance her portfolio each period, the optimal portfolio choice coincides with the (myopic) choice of a one-period investor under the following two sets of conditions:[4]

- the investor has power utility and returns are i.i.d.
- the investor has log utility ($\gamma = 1$) and returns need not be i.i.d. (in fact, this investor will maximize expected log return, and the K-period log return is the sum of one-period returns: therefore, with rebalancing, the sum is maximized by making each period the optimal one-period choice),

- as well understood in the financial literature since the contributions of Samuelson (1969) and Merton (1969, 1971).

Optimality of the myopic strategy can be found also when the investor is concerned with the level of *consumption* in each period (and not only with a terminal value for financial wealth). In this framework, the joint consumption-saving and asset allocation problem is often formulated in an infinite-horizon setting, yielding portfolio rules that depend on preference parameters and state variables, but not on time. The length of the effective investment horizon is governed by the choice of a rate of time preference to discount future utility. With power utility, under the assumption that the investor's consumption to wealth ratio is constant, the consumption capital asset pricing model ($CCAPM$, Hansen and Singleton 1983) implies that (with c denoting log consumption and w log wealth):

$$\begin{aligned} E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2} &= \gamma \, cov_t(r_{t+1}, \Delta c_{t+1}) \\ &= \gamma \, cov_t(r_{t+1}, \Delta w_{t+1}) = \gamma \, \alpha_t \sigma_t^2 \end{aligned}$$

where the second equality is derived from the assumption of a constant consumption-wealth ratio. The optimal share of the risky asset is therefore

[4] If rebalancing is not allowed (as under a "buy and hold" strategy), with i.i.d. returns over time, all mean returns and variances for individual assets are scaled up by the same factor K, and the one-period portfolio solution is still optimal for a K-period investor. This result holds exactly in continuous time and only approximately in discrete time. However, Barberis (2000) shows that if uncertainty on the mean and variance of asset returns is introduced, the portfolio share of the risky asset α_t decreases as the investment horizon lengthens.

the same as in the myopic case:

$$\alpha_t = \frac{E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2}}{\gamma\, \sigma_t^2}$$

(again, this equivalence result is valid also in the case of multiple risky assets). The constant consumption-wealth ratio is justified under i.i.d. returns (implying that there are no changes in investment opportunities over time) or in the special case of log utility ($\gamma = 1$, implying that the income and substitution effects of varying investment opportunities cancel out exactly, leaving the ratio unaffected).

All the above results have been obtained under the asssumption of CRRA, power utility. This formulation is highly restrictive under (at least) one important respect: it links risk aversion (γ) and the elasticity of intertemporal substitution ($1/\gamma$) too tightly, the latter concept capturing the agent's willingness to substitute consumption over time. Epstein and Zin (1989, 1991) adopt a more flexible framework in which scale-independence is preserved but risk aversion and intertemporal substitution are governed by two independent parameters (γ and ψ respectively). The main result is that risk aversion remains the main determinant of portfolio choice, whereas the elasticity of intertemporal substitution has a major effect on consumption decisions but only marginally affects portfolio decisions. With Epstein-Zin preferences, in the case of one risky asset, the premium over the safe asset is given by:

$$E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2} = \theta\, \frac{cov_t(r_{t+1}, \Delta c_{t+1})}{\psi} + (1-\theta)\, cov_t(r_{t+1}, r^p_{t+1})$$

where $\theta = (1-\gamma)/(1-1/\psi)$ and r^p is the continuously compounded portfolio return. The risk premium is a weighted average of the asset return's covariance with consumption divided by ψ (a CCAPM term) and the covariance with the portfolio return (a traditional CAPM term). Under power utility $\theta = 1$ and only the CCAPM term is present. The two conditions for optimal myopic portfolio choice apply in this case as well:

- if asset returns are i.i.d. the consumption-wealth ratio is constant and covariance with consumption growth equals covariance with portfolio return. In this case

  $$E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2} = \gamma\, cov_t(r_{t+1}, r^p_{t+1})$$

  which implies the myopic portfolio rule;

- alternatively, if $\gamma = 1$, then $\theta = 0$ and the risk premium is simply $cov_t(r_{t+1}, r^p_{t+1})$, again implying optimality of the myopic portfolio rule.

Therefore, what is required for optimality of the myopic portfolio choice is a unit relative risk aversion (not a unit elasticity of intertemporal substitution).

2.1.1 Portfolio choice with variations in investment opportunities

The portfolio choice for a long-term investor can importantly differ from the myopic rule when investment opportunities are time-varying. Investment opportunities can vary over time due to variable real interest rates and variable risk premia. Campbell and Viceira (2001, 1999), among others, study the two cases separately, deriving the optimal portfolio policies for an infinite-horizon investor with Epstein-Zin preferences and no labor income.

Preliminarly, following Campbell (1993, 1996) a linear approximation of the budget constraint is derived and the expected risk premium on the risky asset is expressed in terms only of parameters and covariances between the risky return and current and expected future portfolio returns

$$\begin{aligned} E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2} &= \gamma\, cov_t(r_{t+1}, r^p_{t+1}) \\ &\quad +(\gamma - 1)\, cov_t\left(r_{t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r^p_{t+1+j}\right) \end{aligned} \tag{1}$$

where ρ is a constant of linearization and the last term captures the covariance between the current risky return and the revision in expected future portfolio returns due to the accrual of new information between t and $t+1$. Then, (1) can be applied to portfolio choice under specific assumptions on the behavior of returns over time.

If only *variations in the riskless interest rate* are considered, as in Campbell and Viceira (2001), with constant variances and risk premia, then $(E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r^p_{t+1+j} = (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r^f_{t+1+j}$, and, with a single risky asset we have $cov_t(r_{t+1}, r^p_{t+1}) = \alpha_t^2 \sigma_t^2$. From (1) the optimal portfolio weight on the risky asset is then given by

$$\begin{aligned} \alpha_t &= \underbrace{\frac{1}{\gamma}\frac{E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_t^2}{2}}{\sigma_t^2}}_{\text{myopic demand}} \\ &\quad \underbrace{+(1 - \frac{1}{\gamma})\frac{cov_t\left(r_{t+1}, -(E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r^f_{t+1+j}\right)}{\sigma_t^2}}_{\text{intertemporal hedging demand}} \end{aligned} \tag{2}$$

Now, in addition to the asset's risk premium relative to its variance, which determines the "myopic" demand for the asset, a second demand component is relevant (for $\gamma \neq 1$). This component (related to the asset return's covariance with reductions in expected future riskless interest rates, relative to its variance) captures the "intertemporal hedging demand" of Merton (1973), whose weight tends to one as γ increases. The risky asset is held not only for its expected premium, but also because it allows to hedge future expected changes in the portfolio return (due to changes in the riskless rate) compensating the investor for the loss in interest income. This role remains also when risk aversion is increased, and the myopic component of demand tends to zero. On practical grounds, inflation-indexed long-term bonds or, less effectively, nominal long-term bonds can provide this kind of intertemporal hedging, since their returns covary with declines in the level of interest rates, and find their place in the portfolio of long-term investors alongside stocks.

A second empirically relevant case of time-varying investment opportunities involves *variable premia on the risky assets.* Campbell and Viceira (1999) explore the implications of variable risk premia for optimal asset allocation in the case of only one risky asset and constant riskless rate. Here, time-varying investment opportunities are captured by a variable, mean-reverting excess return on the only risky asset. Formally:

$$\begin{aligned} r_{t+1} &= E_t r_{t+1} + u_{t+1} \\ E_t r_{t+1} - r^f + \frac{\sigma_u^2}{2} &= x_t \\ x_{t+1} &= \mu + \phi(x_t - \mu) + \eta_{t+1} \end{aligned}$$

where the state variable x_t summarizes investment opportunities at time t. Innovations u_{t+1} and η_{t+1} may be correlated, with covariance $\sigma_{\eta u}$. This covariance generates intertemporal hedging demand for the risky asset by long-term investors, since it measures the ability of the risky asset to effectively hedge changes in investment opportunities. In fact

$$cov_t(r_{t+1}, x_{t+1}) = cov_t(r_{t+1}, r_{t+2}) = \sigma_{\eta u}$$

so that, in the empirically relevant case $\sigma_{\eta u} < 0$, there is "mean-reversion" in the risky asset return: an unexpectedly high return today reduces expected returns in the future. Under this set of assumptions, the optimal portfolio

share of the risky asset contains the two components, as in (2):

$$\alpha_t = \underbrace{\frac{1}{\gamma}\frac{E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_u^2}{2}}{\sigma_u^2}}_{\text{myopic demand}} + \underbrace{(1-\frac{1}{\gamma})\left(\frac{-\sigma_{\eta u}}{\sigma_u^2}\right)\left[b_1(\mu,\phi) + b_2(\phi)\frac{E_t r_{t+1} - r^f_{t+1} + \frac{\sigma_u^2}{2}}{\sigma_u^2}\right]}_{\text{intertemporal hedging demand}} \quad (3)$$

where $b_1(\mu,\phi)$ is positive and increasing in μ and decreasing in ϕ, and $b_2(\phi)$ is positive and increasing in ϕ. The intertemporal hedging demand is captured by the term involving $\sigma_{\eta u}$: in the empirically relevant case $\sigma_{\eta u} < 0$ (and $\mu > 0$) a sufficiently risk-averse investor ($\gamma > 1$) will hold a larger portfolio share in the risky asset than a myopic one, exploiting the possibility of hedging expected future changes in investment opportunities. Overall, a conservative long-run investor should respond to mean-reverting risky returns by increasing her average portfolio share invested in the risky asset.

2.2 Asset allocation with human wealth

The results for optimal asset allocation mentioned so far apply in the case of fully tradable financial wealth. Adding a typically non-tradable human wealth component (i.e. the expected present discounted value of future labor earnings) is an important step towards the construction of models useful for practical asset allocation strategies of long-term investors such as pension funds. In this case, the analysis above must be suitably adapted, as first done by Bodie, Merton and Samuelson (1992). Here, we briefly consider several cases of increasing complexity.

2.2.1 Baseline case: riskless labour income

The simplest case to analyze is the asset allocation choice of an investor endowed with power utility and a non-stochastic labor income, faced with one riskless and one risky asset. In this case (non-tradable) human wealth H_t is the present discounted value of all future earnings discounted at the riskless rate and is equivalent to the holding of the riskless asset. Therefore, the investor will choose the portfolio share of the risky asset α_t so as to make the nominal holdings of the asset equal to the optimal holdings in the unconstrained case of fully tradable (financial and human) wealth $\hat{\alpha}_t$:

$$\alpha_t = \frac{\hat{\alpha}_t(W_t + H_t)}{W_t} = \frac{E_t r_{t+1} - r^f + \frac{\sigma^2}{2}}{\gamma\sigma^2}\left(1 + \frac{H_t}{W_t}\right) \geq \hat{\alpha}_t \quad (4)$$

Thus, in the presence of riskless non-tradable human wealth, the investor's financial portfolio will be tilted towards the risky asset. The share α_t is increasing in the ratio H_t/W_t and therefore changes over the investor's life cycle for (at least) two reasons: (i) along the life-cycle, H changes relative to W, being higher at the beginning of the working life and lower at retirement; (ii) it changes with financial asset returns: when the risky asset performs well, W increases relative to H and the optimal share of the risky asset decreases, with the investor rebalancing his portfolio away from the risky asset.

More complicated cases are now analyzed, under the simplifying assumptions of power utility, i.i.d. financial asset returns (ruling out time-varying investment opportunities), and no life-cycle perspective (i.e. either a single period or an infinite investment horizon with fixed probability of retirement is considered). We will adopt an explicit life-cycle perspective in the operative model of the next section.

2.2.2 Adding labour income uncertainty

The investor has a one-period horizon (so that no saving decision is involved) and earns a lognormally distributed labour income l_t, potentially correlated with the return on the risky asset: $cov_t(l_{t+1}, r_{t+1}) \equiv \sigma_{lu}$. In this setting, the optimal portfolio rule is given by:

$$\alpha_t = \frac{1}{\rho}\left(\frac{E_t r_{t+1} - r^f + \frac{\sigma_u^2}{2}}{\gamma \sigma_u^2}\right) + \underbrace{\overbrace{\left(1 - \frac{1}{\rho}\right)}^{(-)}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right)}_{\text{labor income hedging demand}} \tag{5}$$

where $\rho = \left(1 + \overline{H/W}\right)^{-1} < 1$ (with $\overline{H/W}$ being the average human to financial wealth ratio) captures the elasticity of consumption to financial wealth and plays a crucial role in linking consumption to the optimal portfolio choice. The first component of the risky asset share in (5) is the optimal share when labour income risk is idiosyncratic (i.e. $\sigma_{lu} = 0$), and corresponds to the simple case of riskless income in (4), confirming the result (since $1/\rho > 1$) that the optimal share of the risky asset is higher than in the absence of labour income, when all wealth is tradable. The second is an income hedging component. The risky asset is desirable if it allows to hedge consumption against low realizations of labor income: if $\sigma_{lu} < 0$ the risky asset is a good hedge and this increases its portfolio share.

2.2.3 Adding flexible labour supply

If labour supply can be flexibly adjusted by the investor, she can compensate for losses on the financial portfolio by increasing work effort: this additional margin of adjustment makes the investor more willing to take on financial risk, as shown by Bodie, Merton and Samuelson (1992). In this extended setting, the optimal share of the risky asset is

$$\alpha_t = \frac{1}{\beta_w}\left(\frac{E_t r_{t+1} - r^f + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) + \overbrace{\left(1 - \frac{1}{\rho}\right)}^{(-)}(1+\nu)\left(\frac{\sigma_{zu}}{\sigma_u^2}\right) \tag{6}$$

where

$$\beta_w = \frac{\rho}{1 + (1-\rho)\gamma\nu}$$

with ν capturing the elasticity of labour supply to the real wage and σ_{zu} measuring the covariance between risky returns and the real wage (as $\nu \to 0$, labour supply becomes infinitely inelastic and $\beta_w \to \rho$ as in the case of fixed labour supply in (5): in all cases $0 \leq \beta_w \leq \rho$). The ability to adjust labour supply increases the risky asset share ($\beta_w \leq \rho$) if wages are uncorrelated with risky returns, and as the elasticity of labour supply increases the portfolio share increases. The sensitivity of the portfolio allocation to a non-zero σ_{zu} is measured by $(1-1/\rho)(1+\nu)$ and becomes increasingly negative as ν increases: investors with flexible labour supply are particularly willing to hedge wage risk, since they respond to fluctuations in real wage by changing their work effort and thus the effects of wage shocks on their labor income are magnified.

2.2.4 Extension to a long-horizon setting

Following Viceira (2001), the investor has an infinite horizon (which makes decision rules time-invariant), with a positive probability of retirement π^r (i.e. a zero-labour income state) each period. The expected time until retirement, $1/\pi^r$, is the effective investor's retirement horizon. After retirement the investor may die with probability π^d in each period, so that $1/\pi^d$ is his expected lifetime after retirement. Labor income is subject only to permanent shocks (a -log- random walk process with drift), so that income growth is:

$$\Delta l_{t+1} = g + \xi_{t+1}$$

with

$$cov_t(r_{t+1}, \Delta l_{t+1}) = cov_t(u_{t+1}, \xi_{t+1}) = \sigma_{u\xi}$$

In each period, the investor can be in either of two states (retired or employed), and the solution to the intertemporal optimization problem depends

on the state. For a *retired* investor the optimal portfolio share of the risky asset, α^r, is simply given by the myopic solution:

$$\alpha_t^r = \frac{E_t r_{t+1} - r^f + \frac{\sigma_u^2}{2}}{\gamma\, \sigma_u^2}$$

For an *employed investor* the (approximate) portfolio share of the risky asset,α^e, is:

$$\alpha_t^e = \frac{1}{\bar{b}_1}\left(\frac{E_t r_{t+1} - r^f + \frac{\sigma_u^2}{2}}{\gamma \sigma_u^2}\right) - \left(\frac{\pi^e(1-b_1^e)}{\bar{b}_1}\right)\left(\frac{\sigma_{u\xi}}{\sigma_u^2}\right) \tag{7}$$

where $0 < b_1^e < 1$ is the elasticity of consumption to financial wealth for an employed investor, and $\bar{b}_1 = \pi^e b_1^e + (1-\pi^e)$ is the average consumption elasticity over the two states (the elasticity in the retirement case being one). Given $b_1^e < 1$, negative shocks to financial wealth do not cause a proportional reduction in consumption, since the employed investor can use labour income to shield consumption from unexpected declines in financial wealth.

The general form of the rule is the same as in the single-period case, with two components, the latter depending on the correlation between labor income and return shocks, thus having the nature of a hedging demand. In fact, when labor income is idiosyncratic ($\sigma_{u\xi} = 0$) only the first component is present, with the average wealth elasticity of consumption to wealth $\bar{b}_1 < 1$. Therefore, the optimal allocation to the risky asset is larger for employed investors than for retired investors. The second term represents the income hedging component of optimal allocation, with a sign opposite to the sign of $\sigma_{u\xi}$ (negative correlation implying that the risky asset is a good hedge against bad labor income realizations).

3 A basic life-cycle model

A fundamental insight from the models surveyed in the preceding section, introducing uninsurable labor income risk in an otherwise standard framework and adopting either a single-period or an infinite investment horizon, is that the optimal asset allocation depends crucially on the ratio of discounted expected future labor income (i.e. human wealth) to accumulated financial wealth. This ratio typically changes over the investor's life cycle in a way that simple assumptions on the stochastic process generating labor income are not capable to capture. Instead, a model with a more realistic age profile of labor income (making human wealth increase relative to financial wealth in the early part of the working life to reach a peak, and then decline in

the years towards retirement) is needed to address the issue of how investors should optimally adjust their financial portfolio over their life cycle.

Adopting an explicit life-cycle perspective, this section presents a model, built mainly on Campbell, Cocco, Gomes and Maenhout (2001), Cocco, Gomes and Maenhout (2005) and Gomes and Michaelides (2004, 2005), that can be used to systematically address portfolio choice over the life cycle.

We do not allow also for excess return predictability and other forms of changing investment opportunities over time, as in Michaelides (2002) and Koijen, Nijman and Werker (2008). While both papers document market timing effects on asset allocations when parameters of the return distributions are known with certainty, there is still considerable debate as to the ex-post value of market timing (De Miguel et al., 2008) and return predictability in general (Goyal and Welch, 2008; Fugazza, Guidolin and Nicodano, 2008) when such parameters are estimated by an asset manager.

3.1 The model

We model an investor that maximizes the expected discounted utility of consumption over her entire life. Though the maximum length of the life span is T periods, its effective lenght is governed by age-dependent life expectancy. At each date t, the survival probability of being alive at date $t+1$ is p_t, the conditional survival probability at t. The investor starts working at age t_0 and retires with certainty at age $t_0 + K$. Investor's i preferences at date t are described by a time-separable power utility function:

$$\frac{C_{it_0}^{1-\gamma}}{1-\gamma} + E_{t_0}\left[\sum_{j=1}^{T}\beta^j\left(\prod_{k=0}^{j-1}p_{t_0+k}\right)\frac{C_{it_0+j}^{1-\gamma}}{1-\gamma}\right]$$

where C_{it} is the level of consumption at time t, $\beta < 1$ is an utility discount factor, and γ is the constant relative risk aversion parameter.[5] We rule out utility derived from leaving a bequest, introduced by Cocco, Gomes and Maenhout (2005). Moreover, we do not model labor supply decisions,

[5] As already mentioned, assuming power utility with relative risk aversion coefficient γ constrains the intertemporal elasticity of substitution to be equal to $1/\gamma$. Moreover, γ also governs the degree of relative "prudence" of the consumer RP, related to the curvature of her marginal utility and measured by

$$RP = -\frac{CU'''(C)}{U''(C)} = 1+\gamma$$

Relative prudence is a key determinant of the consumer's optimal reaction to changes in the degree of income uncertainty.

whereby ignoring the insurance property of flexible work effort (allowing investors to compensate for bad financial returns with higher labor income), as in Gomes, Kotlikoff and Viceira (2008).

3.1.1 Labor and retirement income

Available resources to finance consumption over life cycle derive from accumulated financial wealth and from the stream of labor income. At each date t during the working life, the exogenous labor income Y_{it} is assumed to be governed by a deterministic age-dependent growth process $f(t, \mathbf{Z}_{it})$, and is hitted by both a permanent u_{it} and a transitory n_{it} shock, the latter being uncorrelated across investors. Formally, the logarithm of Y_{it} is represented by

$$\log Y_{it} = f(t, \mathbf{Z}_{it}) + u_{it} + n_{it} \qquad t_0 \leq t \leq t_0 + K \tag{8}$$

More specifically, $f(t, \mathbf{Z}_{it})$ denotes the deterministic trend component of permanent income, which depends on age t and on a vector of individual characteristics $\mathbf{Z}_{it}$, such as gender, marital status, houseohld composition and education. Uncertainty of labor income is captured by the two stochastic processes, u_{it} and n_{it}, driving the permanent and the transitory component respectively. Consistently with the available empirical evidence, the permanent disturbance is assumed to follow a random walk process:

$$u_{it} = u_{it-1} + \varepsilon_{it} \tag{9}$$

where ε_{it} is distributed as $N(0, \sigma_\varepsilon^2)$ and is uncorrelated with the idiosyncratic temporary shock n_{it}, distributed as $N(0, \sigma_n^2)$. Finally, the permanent disturbance ε_{it} is made up of an aggregate component, common to all investors, $\xi_t \sim N(0, \sigma_\xi^2)$, and an idiosyncratic component $\omega_{it} \sim N(0, \sigma_\omega^2)$ uncorrelated across investors:

$$\varepsilon_{it} = \xi_t + \omega_{it} \tag{10}$$

As specified below, we allow for correlation between the aggregate permanent shock to labor income ξ_t and innovations to the risky asset returns.

During retirement, income is certain and equal to a fixed proportion λ of the permanent component of the last working year income:

$$\log Y_{it} = \log \lambda + f\left(t_{0+K}, \mathbf{Z}_{it_{0+K}}\right) + u_{it_{0+K}} \qquad t_0 + K < t \leq T \tag{11}$$

where the level of the replacement rate λ is meant to capture at least some of the features of Social Security systems. Other, less restrictive, modelling strategies are possible. For example, Campbell J. Y., J. Cocco, F. Gomes and P. Maenhout (2001) model a system of mandatory saving for retirement

as a given fraction of the (stochastic) labor income that the investor must save for retirement and invest in the riskless asset, with no possibility of consuming it or borrowing against it. At retirement the value of the wealth so accumulated is transformed into a riskless annuity until death.

3.1.2 Investment opportunities

We allow savings to be invested in a short-term riskless asset, yielding each period a constant gross real return R^f, and in two risky assets, called *stocks* and *bonds*. The risky assets yield stochastic gross real returns R_t^s and R_t^b respectively. We maintain that the investment opportunities in the risky assets do not vary over time and model excess returns of stocks and bonds over the riskless asset as

$$R_t^s - R^f = \mu^s + \epsilon_t^s \tag{12}$$
$$R_t^b - R^f = \mu^b + \epsilon_t^b \tag{13}$$

where μ^s and μ^b are the expected stock and bond premia, and ϵ_t^s and ϵ_t^b are normally distributed innovations, with mean zero and variances σ_s^2 and σ_b^2 respectively. We allow the two disturbances to be correlated, with correlation ρ_{sb}. Moreover, we let the innovation on the stock return be correlated with the aggregate permanent disturbance to the labor income, and denote this correlation by ρ_{sY}.

At the beginning of each period, financial resources available for consumption and saving are given by the sum of accumulated financial weath W_{it} plus current labor income Y_{it}, that we call *cash on hand* $X_{it} = W_{it} + Y_{it}$. Given the chosen level of current consumption, C_{it}, next period cash on hand is given by:

$$X_{it+1} = (X_{it} - C_{it})R_{it}^P + Y_{it+1} \tag{14}$$

where R_{it}^P is the portfolio return

$$R_{it}^P = \alpha_{it}^s R_t^s + \alpha_{it}^b R_t^b + \left(1 - \alpha_{it}^s - \alpha_{it}^b\right) R^f \tag{15}$$

with α_{it}^s, α_{it}^b and $\left(1 - \alpha_{it}^s - \alpha_{it}^b\right)$ denoting the shares of the investor's portfolio invested in stocks, bonds and in the riskless asset respectively. We do not allow short sales and assume that the investor is liquidity constrained, so that the nominal amount invested in each of then three financial assets are $F_{it} \geq 0$, $S_{it} \geq 0$ and $B_{it} \geq 0$ respectively for the riskless asset, stocks and bonds, and the portfolio shares are non negative in each period.

The focus of this paper is on optimal asset allocation and savings until retirement, which however also depend on investment opportunities during

retirement. The simulations presented below concern the case when the pension fund continues to optimally invest the retiree's savings into the same three assets. However, the results concerning asset allocation appear to be qualitatively similar in unreported simulations based on the assumption that retirees invest in the riskless asset only.

3.1.3 Solving the life cycle problem

In this standard intertemporal optimization framework, the investor maximizes the expected discounted utility over life time, by choosing the consumption an the portfolio rules given uncertain labor income and asset returns. Formally, the optimization problem is written as:

$$\max_{\{C_{it}\}_{t_0}^{T-1},\left\{\alpha_{it}^s,\alpha_{it}^b\right\}_{t_0}^{T-1}} \left(\frac{C_{it_0}^{1-\gamma}}{1-\gamma} + E_{t_0}\left[\sum_{j=1}^{T}\beta^j\left(\prod_{k=0}^{j-1}p_{t_0+k}\right)\frac{C_{it_0+j}^{1-\gamma}}{1-\gamma}\right]\right) \tag{16}$$
$$s.t. \quad X_{it+1} = \left(X_{it}-C_{it}\right)\left(\alpha_{it}^s R_t^s + \alpha_{it}^b R_t^b + \left(1-\alpha_{it}^s-\alpha_{it}^b\right)R^f\right) + Y_{it+1}$$

with the labor income and retirement processes specified above and short sales and borrowing constraints.

Given the intertemporal nature of the problem, it can be restated in a recursive form, rewriting the value of the optimization problem at the beginning ot period t as a function of the maximized current utility and of the value of the problem at $t+1$ (Bellman equation):

$$V_{it}\left(X_{it},u_{it}\right) = \max_{\{C_{it}\}_{t_0}^{T-1},\left\{\alpha_{it}^s,\alpha_{it}^b\right\}_{t_0}^{T-1}} \left(\frac{C_{it}^{1-\gamma}}{1-\gamma} + \beta p_t E_t\left[V_{it+1}\left(X_{it_{+1}},u_{it+1}\right)\right]\right) \tag{17}$$

At each time t the value fucntion V_{it} describes the maximized value of the problem as a function of the two state variables, the level of cash on hand at the beginning of time t, X_{it}, and u_{it} the level of the stochastic permanent component of income at beginning of t.

In order to reduce the dimensionality of the original problem, to a problem with one state variable we exploit the homogeneity of degree $(1-\gamma)$ of the utility function, and normalize the entire problem by the permanent component of income u_{it}. Thus, we can rewrite (17) as

$$V_{it}\left(X_{it}\right) = \max_{\{C_{it}\}_{t_0}^{T-1},\left\{\alpha_{it}^s,\alpha_{it}^b\right\}_{t_0}^{T-1}} \left(\frac{C_{it}^{1-\gamma}}{1-\gamma} + \beta p_t E_t\left[V_{it+1}\left(X_{it+1}\right)\right]\right) \tag{18}$$

The problem has no closed form solution, hence the optimal values for consumption and portfolio allocation at each point in time have to be derived

numerically. To this aim, we apply a backward induction procedure and obtain optimal consumption and portfolio rules in terms of the state variable starting form the last (possible) period of life T.

In particular, the solution for period T is trivial, considering that, as we do not allow for positive bequest, it is optimal to consume all the available resources (i.e., $C_{iT} = X_{iT}$) implying that

$$V_{iT}\left(X_T\right) = \frac{X_{iT}^{1-\gamma}}{1-\gamma} \tag{19}$$

The value function at T coincides with the direct utility function over the cash on hand available at the beginning of the period. Then, going backwards, for every period, $t = T-1, T-2, ..., t_0$, and for each possible value of the state variable (the initial level of cash on hand at t) the optimal rules for consumption and the assets' portfolio shares are obtained from the Bellman equation (17) using the grid search method.[6] From the Bellman equation, for each level of the state variable X_{it}, the value function at the beginning of time t, $V_{it}(X_{it})$, is obtained by picking the level of consumption and of portfolio shares that maximizes the sum of the utility from current consumption $U(C_{it})$ plus the discounted expected value from continuation, $\beta p_t E_t V_{it+1}\left(X_{it+1}\right)$. The latter value is computed using $V_{it+1}\left(X_{it+1}\right)$ obtained from the previous iteration. In particular, given $V_{it+1}\left(X_{it+1}\right)$, the expecation term is evaluated in two steps. We use numerical integration perfomed by means of the standard Gaussian Hermite quadrature method to approximate the distribution of shocks to labor income and asset returns. Then, cubic spline interpolation is employed to evaluate the value function at points that do not lie on the state space grid.

3.2 Simulation results

The numerical solution method briefly outlined above yields, for each set of parameters chosen, the optimal policy functions for the level of consumption and the shares of the financial portfolio invested in the riskless asset, stocks and bonds as functions of the level of cash on hand. Using those optimal rules, it is then possible to simulate the life-cycle consumption and asset allocation choices of a large number of agents. In this section, we describe results obtained from this procedure, focussing first on a benchmark case and then presenting extensions along various dimensions.

[6] According to this method, the problem is solved over a grid of values covering the space of the state variables and the controls, to ensure that the solution found is a global optimum.

3.2.1 Calibration

Parameter calibration concerns the investor's preferences, the features of the labor income process during working life and retirement, and the moments of the risky asset returns. To obtain results for a benchmark case, we chose plausible sets of parameters mainly referred to the US and based on Cocco, Gomes and Maenhout (2005) and Gomes and Michaelides (2004, 2005).

The investor begins her working life at the age of 20 and works for (a maximum of) 45 periods (K) before retiring at the age of 65. After retirement, she can live for a maximum of 35 periods until the age of 100. In each period, we take the conditional probability of being alive in the next period p_t from the life expectancy tables of the US *National Center for Health Statistics*. As regards to preferences, we set the utility discount factor $\beta = 0.96$, and the coefficient of relative risk aversion $\gamma = 5$ (capturing an intermediate degree of risk aversion).

The labor income process is calibrated using the estimated parameters for US households with high-school education (but not a college degree) in Cocco, Gomes and Maenhout (2005). The age-dependent trend is captured by a third-order polynomial in age, delivering the typical hump-shaped profile until retirement depicted as the dash-dotted line in Figure 1. After retirement income is a constant proportion λ of the final (permanent) labor income, with $\lambda = 0.68$. The continuous line in the figure protrays the whole deterministic trend $f(t, \mathbf{Z}_{it})$, used in the simulations below, that allows also for other personal characteristics. In the benchmark case, the variances of the permanent and transitory shocks (ε_{it} and n_{it} respectively) are $\sigma_\varepsilon^2 = 0.0106$ and $\sigma_n^2 = 0.0738$; in some of the extensions below we let those parameters vary (to explore the effects of increasing labor income uncertainty) but keep the permanent-transitory ratio roughly constant at the 0.14 level. The riskless (constant) interest rate is set at 0.02, with expected stock and bond premia μ^s and μ^b fixed at 0.04 and 0.02 respectively. The standard deviations of the returns innovations are set at $\sigma_s = 0.157$ and $\sigma_b = 0.08$; in the benchmark case, we fix their correlation at a positive but relatively small value: $\rho_{sb} = 0.2$, a value calibrated on the historical annual correlation in the US and close to the choice of Gomes and Michaelides (2004). Finally, we set $\rho_{sY} = 0$ in the benchmark case, imposing a zero correlation between stock return innovations and aggregate permanent labor income disturbances.

3.2.2 Benchmark results

In all simulations we took cross-sectional averages of 10000 agents over their life cycle. Figure 2 displays the simulation results for the pattern of consump-

tion, labor income and accumulated financial wealth for the working life and the retirement period in the benchmark case. The typical life-cycle profile for consumption is generated. Binding liquidity constraints make consumption closely track labor income until the 35-40 age range, when the consumption path becomes less steep and financial wealth is accumulated at a faster rate. After retirement at 65, wealth is gradually decumulated and consumption decreases to converge to retirement income in the last possible period of life.

Before presenting the age profile of optimal portfolio shares, Figures 3 and 4 display the optimal policy rules for the risky asset shares α_{it}^{s} and α_{it}^{b} as functions of the level of (normalized) cash on hand (the problem's state variable); in each figure the optimal fraction of the portfolio invested in stocks and bonds is plotted against cash on hand for investors of four different ages (20, 30, 55 and 75). The basic intuition that should guide the interpretation of those optimal policies, on which the following simulation results are based, is that labor income is viewed by the investor as an implicit holding of an asset. Although in our setting labor income is uncertain (its process being hit by both permanent and transitory shocks), as long as the correlation of asset returns' innovations and labor income disturbances is not too large, labor income is more similar to the risk-free than to the risky assets;[7] therefore, when the present discounted value of the expected future labor income stream (i.e. human wealth) constitutes a sizeable portion of overall wealth, the investor is induced to tilt her portfolio towards the risky assets. The proportion of human out of total wealth is widely different across investors of different age and is one of the main determinants of their chosen portfolio composition.

Looking at Figures 3 and 4, in the case of an investor of age 75, the certain retirement income acts as a holding of the riskless asset and the relatively poor investors (with a small amount of accumulated wealth and current income) will hold a financial portfolio entirely invested in stocks.[8] Wealthier investors will hold a lower portfolio share in stocks (and increase their holdings of bonds), since for them the proportion of the overall wealth implicitly invested in the riskless asset (i.e. human wealth) is lower. At age 55, the investor has yet a decade of relatively high expected labor income before retirement, and she will tend to balance this implicit holding of a low-risk asset with a financial portfolio more heavily invested in risky stocks than older investors: her optimal policies in Figures 3 and 4 are shifted outwards with respect to the 75-year-old investor for all levels of cash on

[7] We recall here that in our benchmark case, there is a zero correlation between stock return and labor income innovations: $\rho_{sY} = 0$.

[8] The portfolio shares of the risky assets are not defined for extremely low values of cash on hand since the investor (af any age) has no savings in this case.

hand.[9] The same intuition applies to earlier ages, for which the optimal stock and bond policies shift gradually outwards as younger investors are considered. The only exception to this pattern occurs for the very young investors (approximately in the 20-25 age range), for whom the labor income profile is increasing very steeply, making it optimal to hold portfolios more invested in stocks (in the figures, the policy functions shift outward in the 20-25 age range).

On the basis of the optimal investment policies, the mean portfolio shares of stocks and bonds across 10000 agents have been obtained by simulation and plotted in Figure 5(a) against age. The age profiles for stock and bonds are mainly determined by the fact that over the life cycle the proportion of overall wealth implicitly invested in the riskless asset through expected labor incomes varies, being large for young investors and declining as retirement approaches. In fact, younger agents invest their entire portfolio in stocks until approximately the age of 40. Middle-age investors (between 40 and the retirement age of 65) gradually shift the composition of their portfolio away from stocks and into bonds, to reach shares of 60% and 40% respectively at the retirement date. Throughout, the holdings of the riskless asset are kept at a minimum (very often zero); only very young investors keep a small fraction of their portfolio in the riskless asset.

Overall, the popular financial advice of holding a portfolio share of risky stocks equal to 100 minus the investor's age (so that $\alpha^s_{age} = (100 - \text{age})/100$), implying a gradual shift toward bonds over life, is not completely at variance with optimally designed investment policies. However, in the benchmark case above the decumulation of stocks is not linear (as suggested by the simple age-dependent rule, according to which the stock share should be run down from 80% at the age of 20 to reach 35% at retirement). A more rigorous comparison of the optimal investment policy with the simple "age rule" will be provided below.

3.2.3 Sensitivity of mean portfolio shares allocation to labor income risk

To evaluate the robustness of the above results, and to explore the sensitivity of optimal asset allocation to changes in the main parameters of the model, the benchmark case can be modified along a number of dimensions, including varying degrees of risk aversion, different shapes of the labor income process, and different assumptions on the moments of the asset returns'

[9] The step-wise appearence of the policy rules is due to the choice of the grid in the numerical solution procedure. The use of a finer grid would deliver smoother policies, at the cost of additional computing time.

distributions. In this subsection, we focus on two important dimensions (and their interactions), concerning the correlation between stock return innovations and the aggregate permanent shock to labor income (ρ_{sY}), set to zero in the benchmark case, and the variances of the permanent and transitory disturbances driving (the stochastic part of) labor income ($\sigma_\varepsilon^2 = 0.0106$ and $\sigma_n^2 = 0.0738$ in the benchmark case), to capture changes in labor income uncertainty. Figure 5 displays the mean share age profiles for stocks and bonds over the accumulation period, ranging from the beginning of the working life at the age of 20 to the retirement age of 65.

First, we let the stock return innovations be positively correlated with the innovations in permanent labor income. Empirical estimates of this correlation for the US include values not significantly different from zero as in Cocco, Gomes and Michaelides (2005) for households with any level of educational attainment, and the relatively high values reported by Campbell, Cocco, Gomes and Maenhout (2001) and Campbell and Viceira (2002), ranging from 0.33 for households with no high-school education to 0.52 for college graduates. Since our calibration of the labor income process reflects the features of households with high-school education, we choose an intermediate value of $\rho_{sY} = 0.4$, close to the value of 0.37 used by Campbell and Viceira (2002). Figure 5(b) displays the optimal portfolio shares of stocks and bonds when $\rho_{sY} = 0.4$. The general pattern of asset allocation obtained in the benchmark case (Figure 5(a)) is confirmed for middle-aged workers, whereas for younger workers (in the 20-40 age range) optimal portfolio shares differ sharply. In fact, the positive correlation between labor income shocks and stock returns makes labor income closer to an implicit holding of stocks rather than of a riskless asset. Younger investors, for whom human capital is a substantial fraction of overall wealth, are therefore heavily exposed to stock market risk and will find it optimal to offset such risk by holding a relatively lower fraction of their financial portfolio in stocks. This effect decreases as workers get older, determining a gradual increase in the portfolio share of stocks until around the age of 40. Finally, as the retirement age approaches, the size of human capital decreases and the investor shifts her portfolio composition again towards safer bonds; this yields a hump-shaped profile for the optimal share of stocks during working life.

The effects of increasing labor income risk on optimal asset allocation over the working life are portrayed in Figures 5(c) and 5(d). In both sets of simulations we increase the variance of both the permanent and the transitory stochastic components of the labor income process, setting now $\sigma_\varepsilon^2 = 0.0408$ and $\sigma_n^2 = 0.269$, keeping their ratio approximately equal to that used in the benchmark case. Panel (c) plots the results for $\rho_{sY} = 0$ as in the benchmark case, whereas panel (d) shows optimal portfolio shares when $\rho_{sY} = 0.4$. When

there is no correlation between labor income and stock returns the effect of increasing labor income risk is more evident: higher labor income risk reduces the optimal share of stocks in the portfolio at any age. As panel (c) shows, the (average) investor holds a diversified portfolio of risky assets even at a very young age, and starts decumulating stocks and increasing the bond share from the age of around 40. At retirement, the share of stocks is much lower than in the benchmark case, reaching around 0.4, with a correspondingly higher fraction invested in bonds.

A similar effect is detected also in the case of positive correlation between stock returns and labor income shocks ($\rho_{sY} = 0.4$). Comparing the portfolio shares in panel (d) (with high labor income risk) with those in panel (b) (with low income risk), the investor chooses a lower portfolio share of stocks at any age, and at retirement the share of stocks is significantly lower than in the case of reduced labor income risk.

3.2.4 Optimal portfolio shares heterogeneity

So far, we presented simulation results in terms of the average optimal portfolio shares across the investors' population. However, in our framework the presence of idiosyncratic labor income shocks may generate substantial heterogeneity in the pattern of financial wealth accumulation over time, and consequently a potentially wide dispersion of the optimal portfolio shares across individuals of the same age (but with widely different levels of accumulated wealth). The degree of heterogeneity in the optimal asset allocation may be an important element in evaluating the performance of pension funds managing individual accounts, whereby each member's asset allocation is adjusted over time on the basis of age and of the history of individual labor income. For this reason, in exploring the sensitivity of the benchmark results to variations in risk aversion (γ), the replacement ratio (λ),[10] and the correlation between permanent labor income shocks and stock return innovations (ρ_{sY}), we focus on the main features of the whole distribution of optimal portfolio shares across the investors' population: for each age, Figures 6-10 display the median and the 5th and 95th percentiles of the distribution of optimal stock and bond portfolio shares.

In Figure 6, panels (a) and (b) present the distribution of portfolio shares for the benchmark values of risk aversion ($\gamma = 5$), the replacement ratio ($\lambda = 0.68$), and the two values of the labor income-stock return correlation ($\rho_{sY} = 0$ and 0.4) already used in Figure 5(a)-(b). Panel (c) highlights the

[10] We do not analyse changes in retirement age, referring the reader to Bodie, Detemple, Otruba and Walter (2004) who investigate this in a general life-cycle setting with stochastic wage, labour supply flexibility, and habit formation.

role of the correlation between permanent labor income shocks and stock returns by assuming $\rho_{sY} = 1$. Note that even this extreme value for ρ_{sY} does not imply a (counterfactually) high correlation between the stock return innovation and the growth rate of individual labor income, since the latter includes a sizeable idiosyncratic component which is uncorrelated with stock returns.[11]

The results confirm that as ρ_{sY} increases young workers invest less in stocks, gradually raising the share of the riskier asset until the age of 40, to start decumulating towards retirement (Benzoni, Collin-Dufresne and Goldstein 2007, Benzoni 2008); in the case of $\rho_{sY} = 1$ the highest stock share in the financial portfolio never exceeds 80%. In all panels the distribution of portfolio shares is highly heterogeneous due to the presence of idiosyncratic labor income shocks (with the exception of young workers in the case of $\rho_{sY} = 0$, who invest the entire portfolio in stocks to compensate for the relatively riskless nature of their human capital). However, some interesting patterns can be detected. The dispersion among workers decreases as they approach retirement, the more so the higher is the labor income-stock return correlation: as ρ_{sY} increases, the histories of labor incomes and the optimal associated portfolio choices tend to converge over time.

The effects of a high risk aversion ($\gamma = 15$) are explored in Figure 7. As expected, the share of stocks is significantly reduced at all ages and for all values of the labor income-stock return correlation. The hump-shaped pattern of the optimal stock share during working life now appears also in the case of $\rho_{sY} = 0$. In order to assess the effects on optimal asset allocation of the generosity of the first-pillar pension system (whose features are summarized by the level of the replacement ratio λ, set at 0.68 in the benchmark case), Figures 8 and 9 display portfolio shares for two different values of the

[11] In fact, using (8), (9) and (10) we can express the correlation between the growth rate of individual labor income ($\Delta \log Y_{it}$) and the stock return innovation (ε_t^s) in terms of ρ_{sY} and the variances of the aggregate and idiosyncratic labor income shocks as:

$$corr(\Delta \log Y_{it}, \varepsilon_t^s) = \frac{1}{\sqrt{1 + \frac{\sigma_\omega^2 + 2\sigma_n^2}{\sigma_\xi^2}}} \cdot \rho_{sY} < \rho_{sY}$$

Using our benchmark value for $\sigma_n^2 = 0.0738$ and attributing all permanent disturbances to the aggregate component, so that $\sigma_\varepsilon^2 = \sigma_\xi^2 = 0.0106$ (σ_ω^2 being 0), we derive an upper bound for $corr(\Delta \log Y_{it}, \varepsilon_t^s)$:

$$corr(\Delta \log Y_{it}, \varepsilon_t^s) \leq 0.26 \cdot \rho_{sY}$$

Therefore, the values for ρ_{sY} used in our simulations (0.4 and 1) correspond to (relatively low) values for $corr(\Delta \log Y_{it}, \varepsilon_t^s)$ of (at most) 0.10 and 0.26, respectively.

replacement ratio, 0.40 and 0.80 respectively (and for the benchmark risk aversion $\gamma = 5$). When the replacement ratio is 0.40, anticipating relatively low pension incomes, agents choose to save more during their working life, accumulating a higher level of financial wealth. This determines a lower optimal share of stocks at all ages and for all values of the labor income-stock return correlation. Finally, Figure 10 displays asset allocation choices in the case of only partial price indexation of pension income. In our framework, all income flows are expressed in real terms; this amounts to an implicit assumption of full indexation of pension income. Partial indexation is simply modelled as a 2 per cent decrease in the replacement ratio λ from the benchmark value of 0.68 at age 65, to reach 0.34 at year 100. From Figures 6-7 and 8-9 a general pattern emerges as to the dispersion in the portfolio shares, which decreases as the retirement age approaches, and the more so the higher is the risk aversion parameter and the lower is the replacement ratio. Indeed, the higher the risk aversion and the lower the replacement ratio, the higher is saving and the the larger is the accumulation of financial wealth over the working life; this, according to the policy functions shown in Figure 3, implies a reduced sensitivity of portfolio composition to the level of human capital. This insensitivity is stronger the closer is the worker to the retirement age, when financial wealth reaches the maximum level.

3.3 Welfare costs of suboptimal asset allocations

Tailoring asset allocations to the specificities of workers' income stories may involve considerable management fees that are not included in our model. To practically assess the welfare gains from optimal asset allocation relative to simpler alternative investment strategies, we present in Table 1 the welfare gains of the optimal strategy computed as the yearly percentage increase in consumption granted by the optimal asset allocation. The first alternative strategy is an "age rule", whereby the risky portfolio share is set at (100-age)% and equally allocated between stocks and bonds. This mirrors the empirical relationship between the average proportion invested in stocks and the fund's horizon for Target Date Funds, which is approximately linear with a slope of -1. Bodie and Treussard (2008) adopt another variant of this formula: starting the process of saving for retirement 40 years before the target retirement date, they set the initial proportion invested in equity to 80% letting it fall to 40% at the target date. Thus the formula for the equity percentage T years from the target date is $40 + T$. The second alternative strategy fixes portfolio shares at 1/3 for each financial asset in our model: this mirrors the $1/N$ rule of DeMiguel, Garlappi and Uppal (2008), that systematically outperforms several optimal asset allocation strategies in *ex*

post portfolio experiments.

The table shows the welfare cost of each sub-optimal strategy for the two values of risk aversion ($\gamma = 5$ and 15) and the three values of ρ_{sY} considered in the simulations above. For each parameter combination, the table reports the mean welfare cost for the overall population and the welfare costs corresponding to the 5th, 50th and 95th percentiles of the distribution of accumulated financial wealth at age 65.

Several results stand out. First, the magnitude of the mean welfare costs is broadly in the range of 1-3%, consistently with Cocco, Gomes and Maenhout (2005). Second, welfare costs fall as risk aversion increases, because high risk aversion implies reduced optimal exposure to the stock market, and risky asset in general. Looking at the cost distribution conditional on wealth, welfare costs increase as financial wealth falls, because a high human to financial wealth ratio implies a relatively high optimal exposure to the stock market. Third, higher welfare costs are associated to lower values of the labor income-stock return correlation, due to the more important role of the stock market in hedging background risk.

Last but not least, the $1/N$ strategy performs consistently better than the "age rule", showing lower mean welfare costs for all parameter combinations. Tabulated results suggest that an unconditional $1/N$ asset allocation is likely to be cost efficient for high wealth worker in medium to high replacement ratio countries. Note, indeed, that the $1/N$ rule implies a reduction of 49 basis points per annum in terms of equivalent consumption for a highly risk averse worker with median wealth and intermediate labor income correlation with stock returns. According to Blake (2008), the annual fee for active portfolio management charged by pension funds ranges between 20 to 75 basis point per year depending on assets under management. Thus, the reduction in welfare (as measured by equivalent consumption flow) due to a sub-optimal asset allocation is lower than the maximum management fee. Moreover, such reduction refers to the benchmark of an optimal asset allocation chosen by an investor who knows precisely the distribution of both labor income and asset returns. On the contrary, asset managers typically make mistakes when estimating the parameters of such distributions, a fact that explains why an equally wighted, $1/N$ allocation usually outperforms optimal strategies in ex post experiments. Individual accounts are instead likely to be cost efficient for low-wealth workers, especially in low replacement ratio countries.

4 Welfare Ratios for Performance Evaluation

Standard methods for evaluating defined contribution pension funds are similar to those used for measuring mutual funds performance. Existing studies often examine the performance of delegated fund managers, which justifies the practice of using the same method. Performance evaluation is based either on the return of the managed portfolio relative to that of an appropriate benchmark or directly on portfolio holdings (see Ferson and Khang, 2002). The investor horizon is usually assumed to be short, and when it is relatively long, as in Blake, Lehmann and Timmermann (1999) the question being asked concerns whether performance is due to strategic asset allocation, as opposed to short-term market timing and security selection. Rarely do studies assess performance at the pension plan level. Recently, Bauer and Frehen (2008) manage to evaluate US pension funds plans against their internal benchmark portfolios.[12]

In principle return-based performance evaluation is appropriate also if the worker's preferences are defined over consumption and there are non-traded assets, as in our life-cycle model. The benchmark portfolio must however be the optimal portfolio for hedging fluctuations in the intertemporal marginal rates of substitution of any marginal investor. On the contrary, the chosen benchmarks typically reflect the state of empirical asset pricing and constraints on available data (Lehmann and Timmermann, 2008). Thus, standard performance evaluation practice relies on the idea that a higher return-to-risk differential maps into better performance, overlooking the pension fund ability in hedging labor income risk and pension risk of plan participants.

Computing the optimal life-cycle asset allocation allows to evaluate pernsion funds' performance with reference to a benchmark that explicitly accounts for the pension plan's role in smooting consumption risk. For instance, we can take the ratio of the worker's *ex-ante* maximum welfare under optimal asset allocation, $V_0\left(R_{it}^{P*}\right)$, by her welfare level under the actual pension fund asset allocation, $V_0\left(R_{it}^{PF}\right)$:

$$WR1 \equiv \frac{V_0\left(R_{it}^{P*}\right)}{V_0\left(R_{it}^{PF}\right)} \tag{20}$$

where R_{it}^{P*} and R_{it}^{PF} are the optimal and actual portfolio return - net of

[12] Elton, Gruber and Blake (2006) investigate whether 401(k) plans offer their participants appropriate investment opportunities such that they can span the frontier generate by an adequate set of alternative investment choices.

management costs - for member i at time t.[13] More precisely, R_{it}^{PF} are simulated returns which are extracted from the estimated empirical distribution of pension fund returns. Similarly, $V_0\left(R_{it}^{PF}\right)$ results by simulation of optimal consumption and savings decisions for pension members, without optimizing for the asset allocation. The higher the value of the welfare ratio $WR1$, the worse the pension fund performance. Importantly, a lower ratio may be due not only to a higher return per unit of financial risk earned by the pension fund, but also to a better matching between the pension fund portfolio and its members' labor income and pension risks.

Table 2 displays welfare ratios $WR1$ computed for various combinations of risk aversion, the replacement ratio, and the correlation between shocks to labor income and stock returns. In the table, it is assumed that the fund follows a suboptimal strategy (the age rule) that is insensitive to members' incomes and replacement ratios, yielding a Sharpe ratio equal to 0.34. The average Sharpe ratio of the optimal rule is consistently lower, from a minimum of 0.24 for $\lambda = 0.8$ and $\rho_{sY} = 0$ to a maximum of 0.31 for $\lambda = 0.4$ and $\rho_{sY} = 1$. Thus, performance evaluated according to a standard return-to-risk metric is worse for the optimal than for the age rule. The picture changes when we look at the proposed welfare metric, that always exceeds 1 -indicating a higher welfare associated with the optimal asset allocation. We can also note that the value of pension funds in smoothing consumption risk is higher the lower are both the member's income and the country's replacement ratio. In fact the higher values for the welfare ratio (1.1) obtain for the fifth income percentile and $\lambda = 0.40$ or 0.68. Such figures are associated to $\rho_{sY} = 0$, i.e. a case where the low correlation between income and stock returns allow for a better hedging of labour income shocks.

Another property of this metric is that it allows for cross-country performance comparisons, along the lines of Antolin (2008), even if countries differ in labor income profiles, replacement ratios, inflation protection for pension annuities and life expectancy. These parameters enter both the numerator and the denominator of $WR1$; thus the cross-country distribution of this ratio is only affected by how well pension funds perform their consumption smoothing role.

It is well known that the investible asset menu in certain countries is restricted by regulation (see Antolin, 2008). If this is the case, the numerator of the welfare ratio ought to be computed conditional on the country investable asset menu so as to evaluate the pension fund manager's ability. If the regulator wants to assess the costs from restricting the asset menu

[13] Estimated management fees ought to be subtracted from portfolio returns when computing workers wealth accumulation.

for retirees, then the relevant ratio should be computed as appropriate ratio could be calculated:

$$WR2 \equiv \frac{V_0\left(R_{it}^{P*}\right)}{V_0\left(R_{it}^{P*restricted}\right)} \tag{21}$$

where the optimal asset allocation enters both the numerator and the denominator, but the asset menu at the numerator is an internationally available one.

The previous section argues that the $1/N$ strategy dominates the optimal asset allocation when the costs of tailoring the asset allocation to workers' profiles exceed their benefits, i.e. the differential in management fees is sufficiently high. In such a case, it could be appropriate to substitute the numerator with the *ex ante* welfare achieved when portfolio returns are associated with the $1/N$ strategy,[14] obtaining

$$WR3 \equiv \frac{V_0\left(R_t^P(1/N)\right)}{V_0\left(R_{it}^{PF}\right)} \tag{22}$$

5 Conclusions

Modern finance theory suggests that the features of the labor income stream over the investor's life cycle are a crucial determinant of her optimal investment policy. Building models that incorporate those characteristics and generate realistic patterns of life-cycle consumption and wealth accumulation can then be an important advance for the design of more appropriate investment policies of defined-contribution pension funds.

In this paper, we make an effort in this direction by setting up an operative framework for analysis (as in Campbell, Cocco, Gomes and Maenhout 2001, Cocco, Gomes and Maenhout 2005 and Gomes and Michaelides 2004, 2005), where the investor faces uncertain labor income during her working life and certain retirement income afterwards. The labor income process is calibrated to capture a realistic hump-shaped pattern with both permanent and transitory shocks. The available financial assets include a riskless short term asset, a high risk premium asset and a low risk premium asset, with potentially correlated returns.

The calibrated version of the model uses US stock index and bond index returns. However, any pair of assets (or baskets of assets) can be accomodated, to the extent that their mean returns, their variances and covariances can be estimated precisely. Returns on foreign assets ought to be expressed in

[14] In general, the numerator ought to be associated with the best suboptimal strategy, taking into account management costs.

foreign currency, with currency risk fully hedged, since there is no explicit dynamics of the exchange rate in this simple version of the model. Furthermore, the model can be used in its current version in economies where inflation is not highly volatile, as the model assumes constant inflation.

A first set of basic results on the optimal asset allocation over the investor's life cycle is presented and a number of extensions are explored. These allow for changes in the generosity of the first-pillar social security provisions (also with incomplete price level indexation), in the degree of investors' risk aversion, the amount of labor income risk as well as in the correlation between labor income shocks and innovations in stock returns. Further, the paper computes and discusses welfare costs associated with the adoption of simple sub-optimal strategies ("age rule" and " $1/N$ rule"). Finally, we discuss welfare-based metrics for pension fund evaluation.

The results on asset allocation sensitivity to changes in labor income profiles suggest that pension plans ought to offer different investment options for workers who require heterogeneous asset allocations. It is however possible to evaluate the performance and the associated participants' welfare costs of simple rules - more easily implementable by pension funds - that partially account for the heterogeneity of optimal portfolio shares, e.g. by grouping members into age classes and applying the optimal "median" share to all members in a specified class.

Further research may be carried out along various directions. First, the asset allocation simulations presented here overlook health shocks during retirement years, which may abruptly reduce disposable income net of healthcare expenses in countries without complete public coverage. One way to address this problem in future implementations is to reduce the yearly pension annuity by the cost of a complementary health-insurance policy, which is likely to be rising in age along with morbility probability. Moreover, future research may further scrutinize our conclusion that the $1/N$ portfolio strategy is likely to be cost efficient for both high wealth and highly-risk-averse-average-wealth workers in medium-to-high replacement ratios countries.

References

[1] Antolin P. (2008) "Pension Fund Performance", OECD WP on Insurance and Private Pensions, n. 20

[2] Barberis N. (2000) "Investing for the Long-Run when Returns are Predictable", *Journal of Finance*, 56, 1247-1292,

[3] Bauer R., and R.G.P. Frehen (2008) The Performance of US Pension Funds, www.ssrn.com

[4] Benzoni L., (2008) "Investing over the Life Cycle with Long-Run Labor Income Risk", mimeo

[5] Benzoni L., P. Collin-Dufresne and R.G. Goldstein (2007) "Portfolio Choice over the Life-Cycle when the Stock and Labor Markets are Cointegrated", *Journal of Finance*, 62, 5, 2123-2167

[6] Blake D. (2008) "It is all back to front: critical issues in the design of defined contribution pension plans", in Broeders D., S. Eijffinger and A. Houben (eds.), *Frontiers in Pension Finance*, Edward Elgar

[7] Blake D., D. Lehmann D. and A. Timmermann (1999), "Asset allocation dynamics and pension fund performance", *Journal of Business,* 72

[8] Bodie Z., J.B. Detemple, S.Otruba and S.Walter (2004) "Optimal consumption-portfolio choices and retirement planning", *Journal of Economic Dynamics and Control*, 28(6), 1115-1148

[9] Bodie Z., R.C. Merton and W. Samuelson (1992) "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model", *Journal of Economic Dynamics and Control*, 16, 427-449

[10] Bodie, Zvi and J. Treussard (2008) "Making Investment Choices as Simple as Possible: An Analysis of Target Date Retirement Funds", mimeo

[11] Cairns A.J.G., D. Blake and K. Dowd (2006) "Stochastic Lifestyling: Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans", *Journal of Economic Dynamics and Control*, 30, 843-877

[12] Campbell J.Y. (1993) "Intertemporal Asset Pricing without Consumption Data", *American Economic Review*, 83, 487-512

[13] Campbell J.Y. (1996) "Understanding Risk and Return", *Journal of Political Economy*, 104, 298-345

[14] Campbell J. Y., J. Cocco, F. Gomes and P. Maenhout (2001) "Investing Retirement Wealth: a Life-Cycle Model", in Campbell J.Y. and M. Feldstein (eds.), *Risk Aspects of Investment-Based Social Security Reform*, University of Chicago Press

[15] Campbell J. Y. and L. Viceira (1999) "Consumption and Portfolio Decisions when Expected Returns are Time Varying", *Quarterly Journal of Economics*, 114, 433-495

[16] Campbell J. Y. and L. Viceira (2001) "Who Should Buy Long-Term Bonds?", *American Economic Review*, 91, 99-127

[17] Campbell J. Y. and L. Viceira (2002) *Strategic Asset Allocation: Portfolio Choice for Long-Tem Investors*, Oxford University Press

[18] Cocco J. (2004) Portfolio Choice in the presence of Housing, *Review of Financial Studies*, 535-567

[19] Cocco J., F. Gomes and P. Maenhout (2005) "Consumption and Portfolio Choice over the Life Cycle", *Review of Financial Studies*, 18, 491-533

[20] DeMiguel V., L. Garlappi and R. Uppal (2008) "Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?", *Review of Financial Studies*, forthcoming

[21] Elton, J. E., M. J. Gruber, and C. R. Blake, 2006, Participant reaction and the performance of funds offered by 401(k) plans, EFA Zurich Meetings

[22] Epstein L. and S. Zin (1989) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework", *Econometrica*, 57, 937-969

[23] Epstein L. and S. Zin (1991) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Investigation", *Journal of Political Economy*, 99, 263-286

[24] Ferson W. and K. Khang (2002) "Conditional Performance Measurement Using Portfolio Weights: Evidence for Pension Funds," *Journal of Financial Economics*

[25] Fugazza C., M. Guidolin and G. Nicodano (2008) "Time and Risk Diversification in Real Estate Investments: Assessing the Ex Post Economic Value", Netspar working paper

[26] Gomes F.J., L.J. Kotlikoff and L.M. Viceira (2008) "Optimal Life-Cycle Investing with Flexible Labor Supply: A Welfare Analysis of Life-Cycle Funds", *American Economic Review*, 98, 2, 297-303

[27] Gomes F.J. and A. Michaelides (2004) "A Human Capital Explanation for an Asset Allocation Puzzle", mimeo

[28] Gomes F.J. and A. Michaelides (2005) "Optimal Life-Cycle Asset Allocation:Understanding the Empirical Evidence", *Journal of Finance*, 60, 869-904

[29] Goyal, A. and I. Welch (2008) "A Comprehensive Look at the Empirical Performance of Equity Premium Prediction", *Review of Financial Studies,* 21 (4): 1453-1454

[30] Hansen L.P. and K.J. Singleton (1983) "Stochastoc Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns", *Journal of Political Economy*, 91, 249-268

[31] Jagannathan R. and N.R. Kocherlakota (1996) "Why Should Older People Invest Less in Stocks Than Younger People?", *Federal Reserve Bank of Minneapolis Quarterly Review*, 11-23

[32] Koijen R.S.J, T. Nijman and B.J.M. Werker (2006) "Optimal annuity risk management", Netspar Discussion Paper

[33] Koijen R.S.J, T. Nijman and B.J.M. Werker (2008) "When Can Life-Cycle Investors Benefit from Time-Varying Bond Risk Premia?", SSRN working paper

[34] Lehmann, B. and A. Timmermann (2008) "Performance Measurement and Evaluation" in A.Boot and A.Thakor "*Handbook of Financial Intermediation and Banking*"

[35] Merton R.C. (1969) "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case", *Review of Economics and Statistics*, 51, 247-257

[36] Merton R.C. (1971) "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", *Journal of Economic Theory*, 3, 373-413

[37] Merton R.C. (1973) "An Intertemporal Capital Asset Pricing Model", *Econometrica*, 41, 867-887

[38] Michaelides A. (2002), "Portfolio Choice, Liquidity Constraints and Stock Market Mean Reversion", SSRN working paper

[39] OECD (2007), "Pensions at a Glance - Public Policies Across OECD Countries", OECD Publishing

[40] Samuelson P.A. (1969) "Lifetime Portfolio Selection by Dynamic Stochastic Programming", *Review of Economics and Statistics*, 51, 239-246

[41] Viceira L. (2001) "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income", *Journal of Finance*, 56, 433-470

Table 1 - Welfare Costs
Mean welfare costs for the overall population are reported.
Moreover, we report welfare gains corresponding to percentiles of financial wealth accumulated at age 65

	Risk aversion 5		Risk aversion 15	
		$\rho_{sy}=0$		
	WelfareCosts (100-age)/2	***1/3***	***WelfareCosts (100-age)/2***	***1/3***
Mean	0.021	0.018	0.015	0.013
5th percentile	0.032	0.027	0.043	0.037
50thpercentile	0.024	0.021	0.012	0.0105
95thpercentile	0.004	0.003	0.004	0.004
		$\rho_{sy}=0.4$		
	WelfareCosts (100-age)/2	***1/3***	***WelfareCosts (100-age)/2***	***1/3***
Mean	0.012	0.012	0.012	0.011
5th percentile	0.027	0.022	0.036	0.034
50thpercentile	0.013	0.011	0.006	0.005
95thpercentile	0.001	0.001	0.002	0.001
		$\rho_{sy}=1$		
	WelfareCosts (100-age)/2	***1/3***	***WelfareCosts (100-age)/2***	***1/3***
Mean	0.012	0.011	0.011	0.010
5th percentile	0.025	0.023	0.032	0.031
50thpercentile	0.012	0.011	0.005	0.005
95thpercentile	0.002	0.001	0.002	0.002

Table 2 - Welfare Ratios
Mean welfare ratio for the overall population are reported.
Moreover, we report welfare ratios corresponding to percentiles of financial wealth accumulated at age 65.

	Risk aversion 5		
Replacement ratio	**0.68**	**0.4**	**0.8**
		$\rho_{sy}=0$	
Sharpe ratio			
Optimal	0.260	0.286	0.244
Age rule	0.337	0.337	0.337
Welfare Ratio			
Mean	1.051	1.096	1.044
5th percentile	1.101	1.096	1.048
50thpercentile	1.056	1.074	1.057
95thpercentile	1.014	1.011	1.007
		$\rho_{sy}=0.4$	
Sharpe ratio			
Optimal	0.273	0.310	0.257
Age rule	0.337	0.337	0.337
Welfare Ratio			
Mean	1.033	1.049	1.028
5th percentile	1.049	1.076	1.028
50thpercentile	1.040	1.054	1.032
95thpercentile	1.009	1.012	1.007
		$\rho_{sy}=1$	
Sharpe ratio			
Optimal	0.296	0.314	0.264
Age rule	0.337	0.337	0.337
Welfare Ratio			
Mean	1.025	1.037	1.020
5th percentile	1.042	1.029	1.032
50thpercentile	1.031	1.043	1.021
95thpercentile	1.002	1.009	1.002

Figure 1 Labor income process
The figure reports the fitted polynomial in age and personal characteristics derived according Cocco et al. (2005) calibrations for households with high school education.

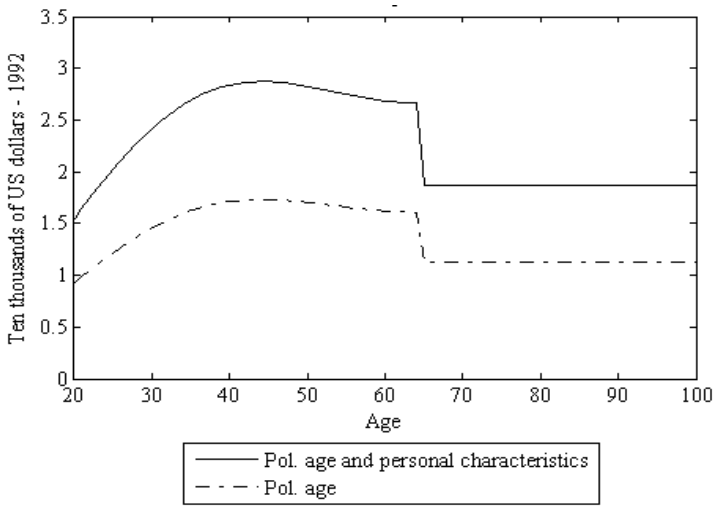

Figure 2 Life cycle profiles of consumption, income and wealth
The figure reports simulated consumption, income and wealth profile for the benchmark case.

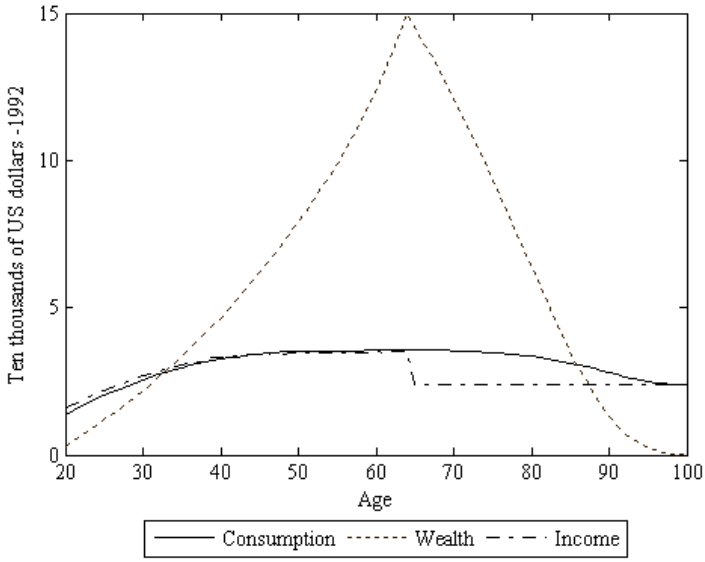

Figure 3 Policy functions
The figure reports the policy functions for the portfolio shares invested in stocks at different ages.

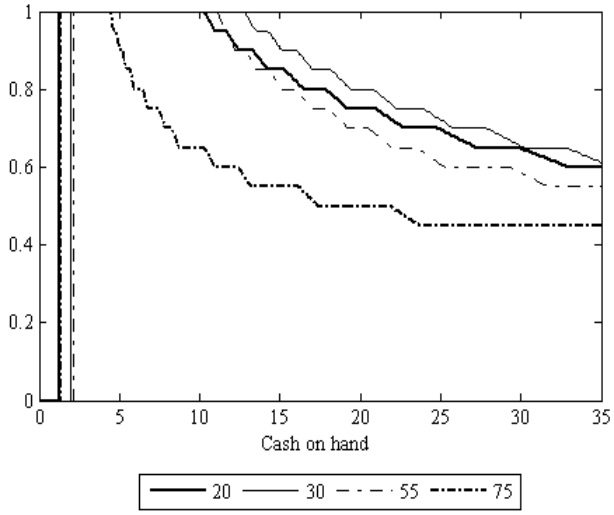

Figure 4 Policy functions
The figure reports the policy functions for the portfolio shares invested in bonds at different ages.

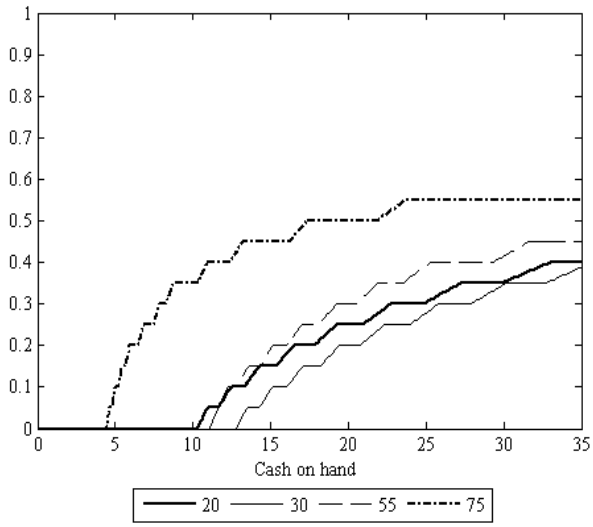

Figure 5

The figure reports mean share profiles, as a function of age, for stocks and bonds. The replacement ratio is equal to 0.68, the correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income is 0 and 0.4. The variance of permanent and transitory shocks in the benchmark case is 0.0106 and 0.0738 in panels (a) and (b), while it is 0.0408 and 0.269 in panels (c) and (d).

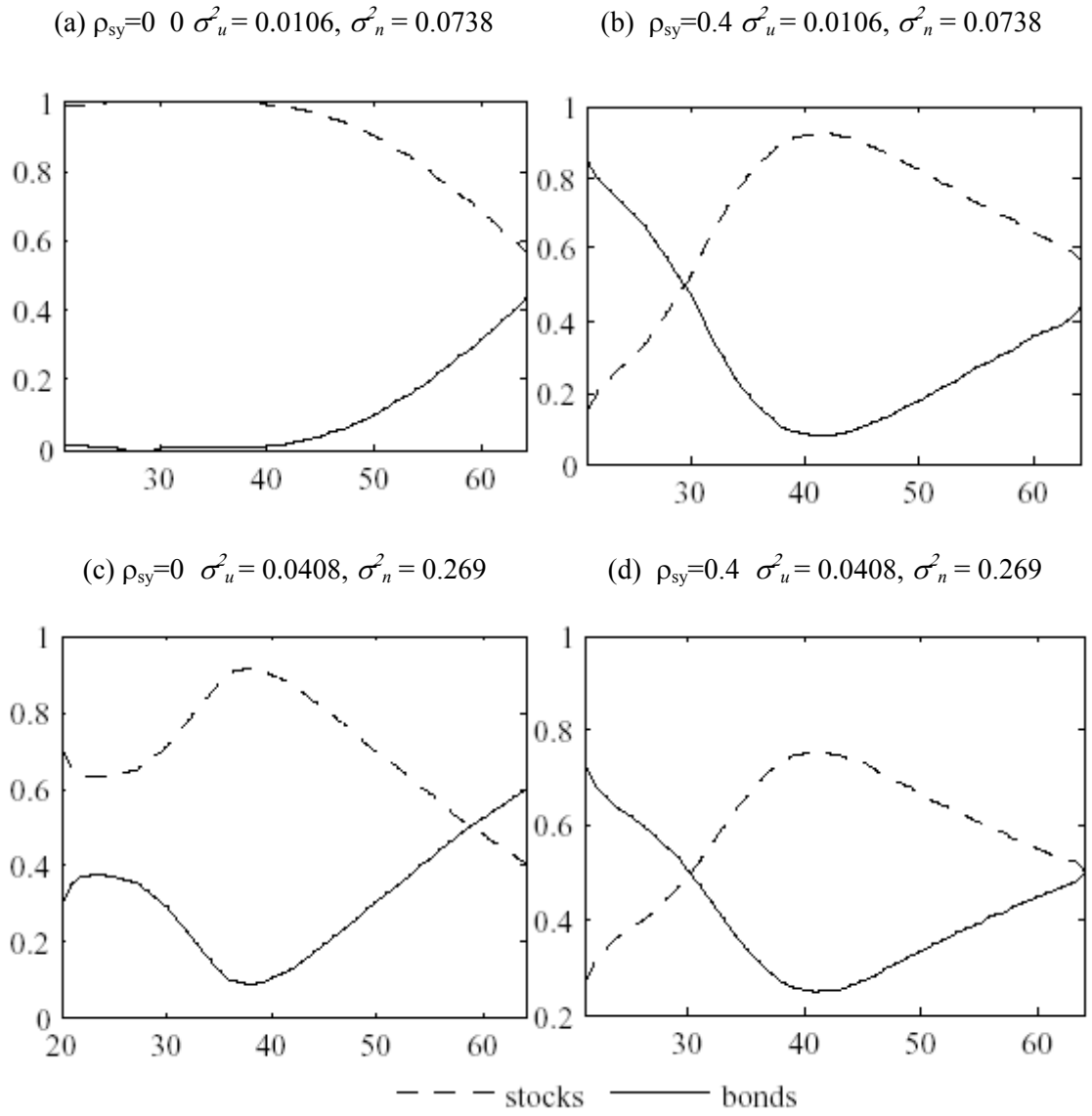

Figure 6 - Low risk aversion (γ=5)

This figure reports share profiles, as a function of age, for stocks and bonds. The solid line represents the shape of the median portfolio share, while the (dotted) dashed refer to the (5th) 95th percentiles. The replacement ratio is equal to 0.68, the correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income varies between 0 and 1.

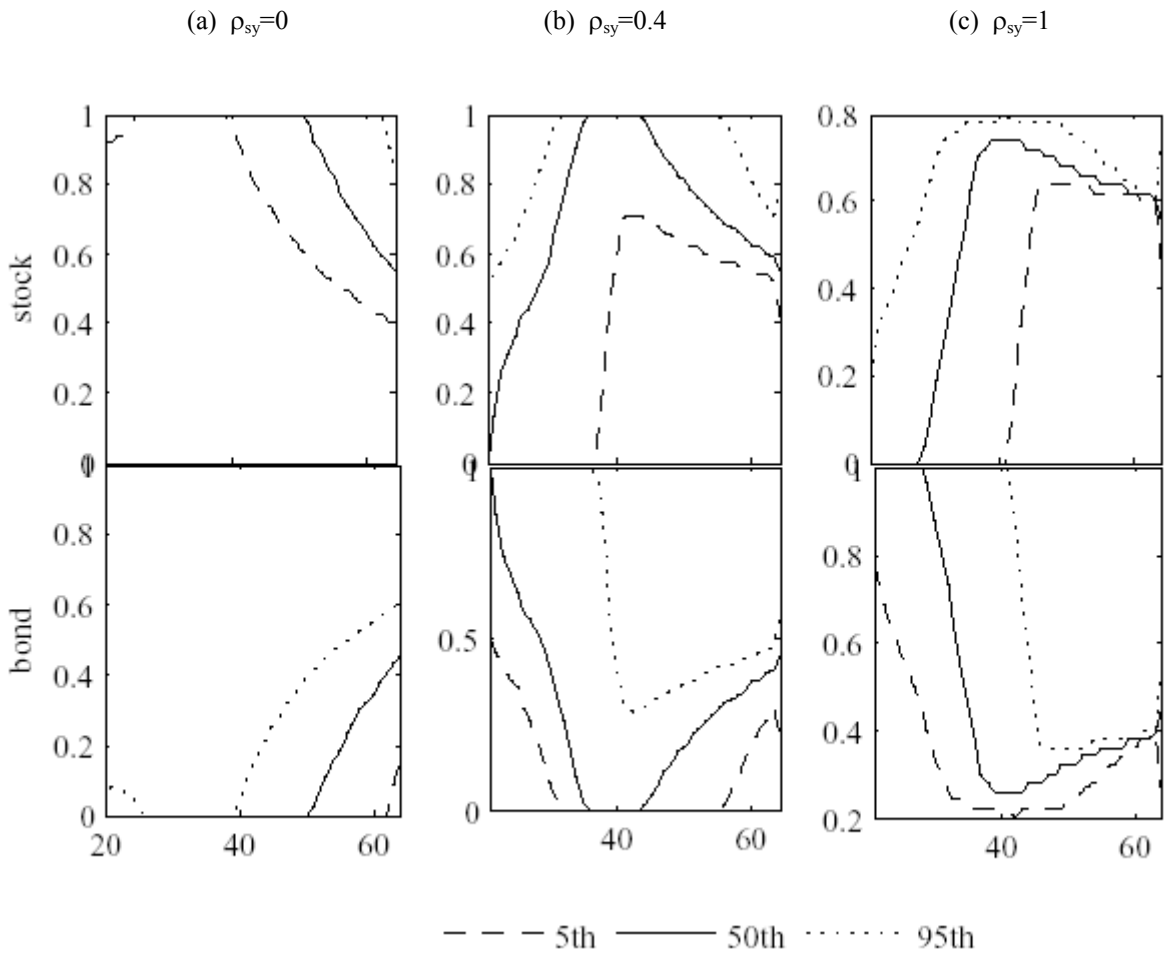

Figure 7 - High risk aversion (γ=15)

This figure reports share profiles, as a function of age, for stocks and bonds. The solid line represents the shape of the median portfolio share, while the (dotted) dashed refer to the (5th) 95th percentiles. The replacement ratio is equal to 0.68, the correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income varies between 0 and 1.

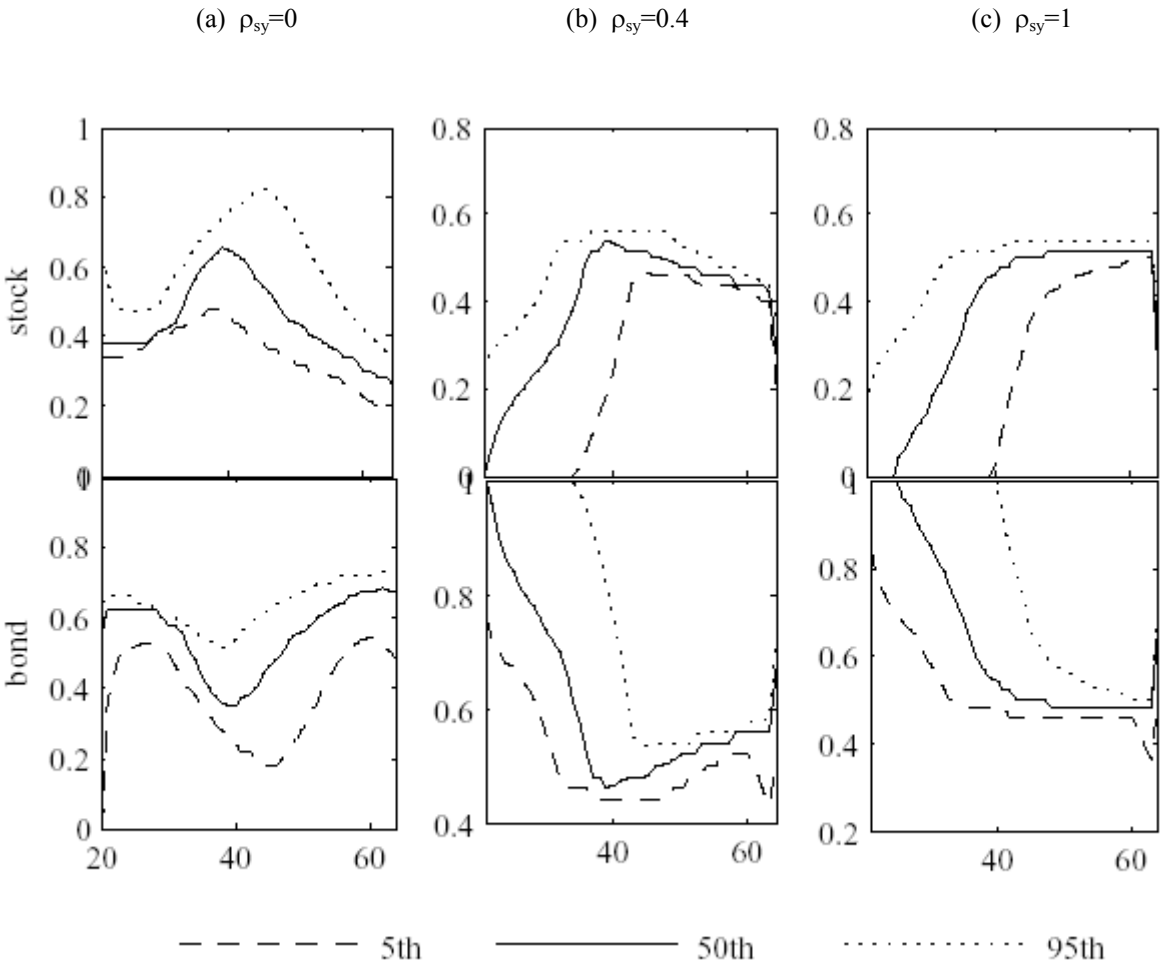

Figure 8 - Low replacement ratio (λ=0.40)

This figure reports share profiles, as a function of age, for stocks and bonds. The solid line represents the shape of the median portfolio share, while the (dotted) dashed refer to the (5th) 95th percentiles. The replacement ratio is equal to 0.40 ,the correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income varies between 0 and 1

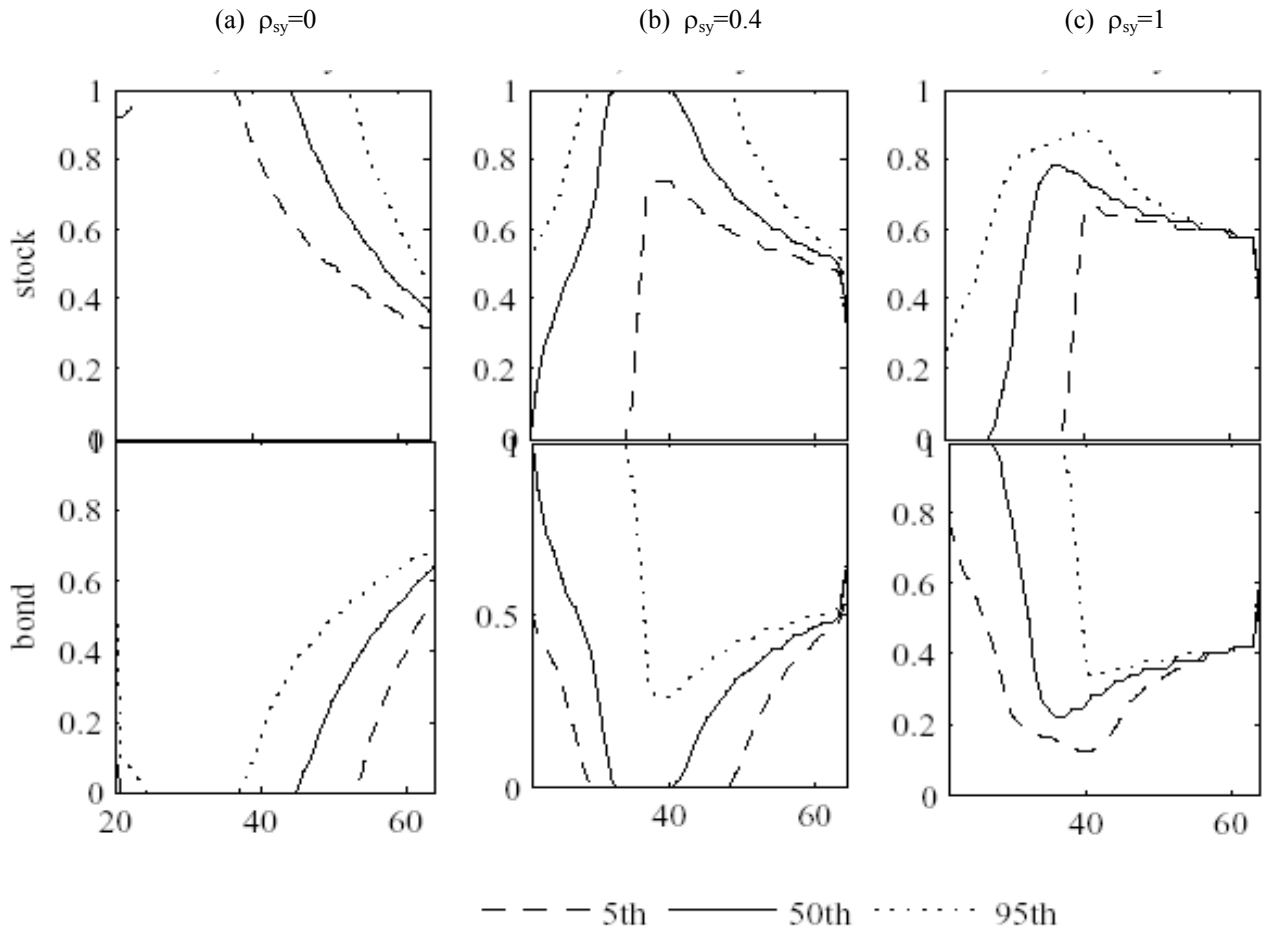

Figure 9 - High replacement ratio (λ=0.80)

This figure reports share profiles, as a function of age, for stocks and bonds. The solid line represents the shape of the median portfolio share, while the (dotted) dashed refer to the (5th) 95th percentiles. The replacement ratio is equal to 0.80, the correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income varies between 0 and 1.

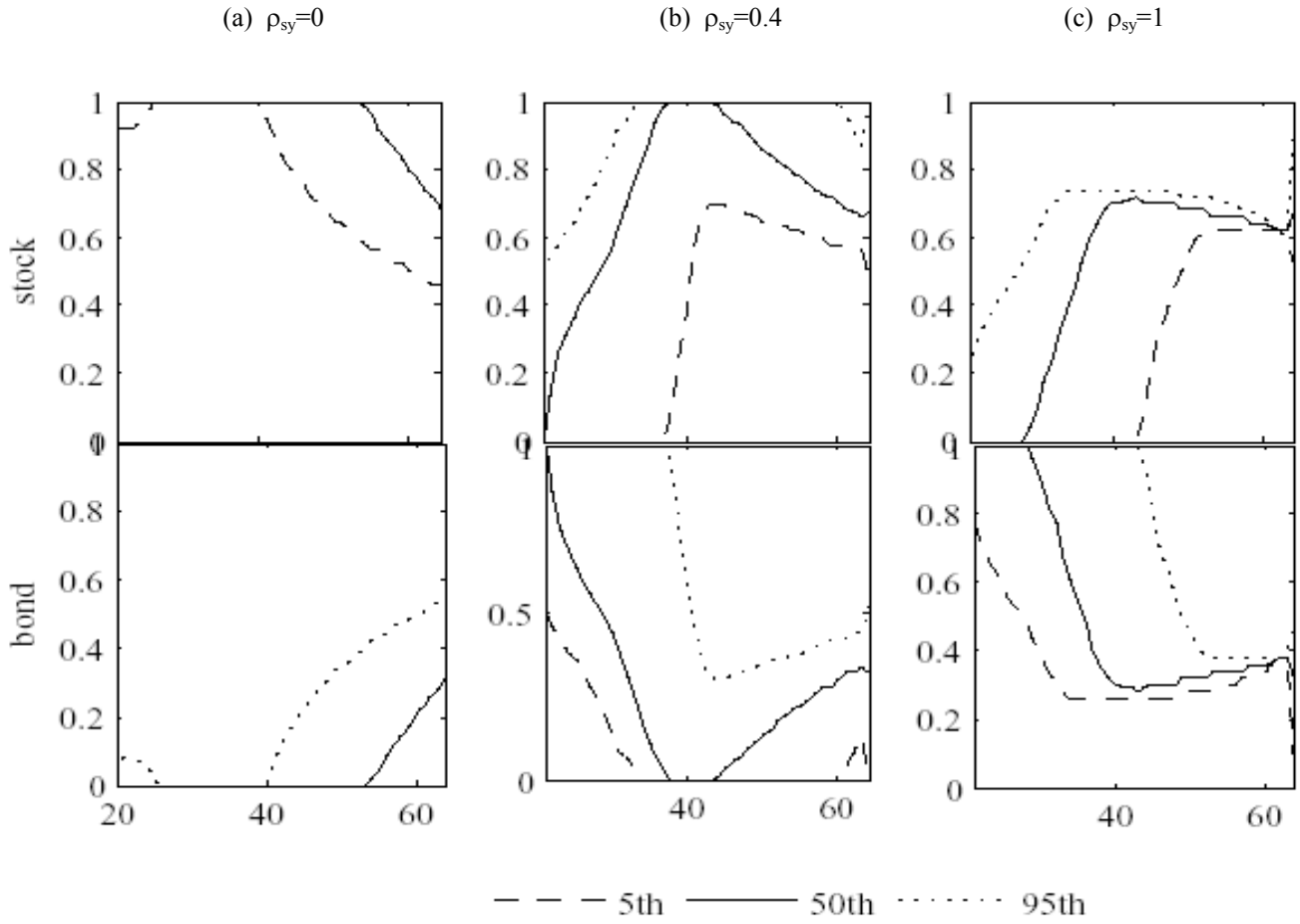

Figure 10 - Declining replacement ratio

This figure reports share profiles, as a function of age, for stocks and bonds. The solid line represents the shape of the median portfolio share, while the (dotted) dashed refer to the (5th) 95th percentiles. pension treatments at the beginning of retirement correspond to a replacement ratio of 0.68 and then decrease by 0.02 per year due to inflation (which implies an average replacement ratio of about 0.45 over all retirement ages). The correlation between stock and bond returns is set to 0.2 while the one between stocks and labour income varies between 0 and 1.