Annual and Quarterly Productivity Measures

Bert M. Balk*
Erasmus Research Institute of Management
Erasmus University Rotterdam
and
Statistics Netherlands
P. O. Box 4000, 2270 JM Voorburg, The Netherlands
E-mail bblk@cbs.nl

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Abstract

Productivity change is generically measured in index form as ratio of output quantity index over input quantity index. The two periods compared by such an index are usually years, since in the majority of cases the most natural accounting period appears to be a year.

Now there seems to be a demand from policy side for more frequent statistics, for example quarterly. This paper explores, from a theoretical perspective, the options for obtaining consistency between annual and quarterly (or more general: between period and subperiod) measures of productivity change.

Keywords: Productivity index; annual; quarterly.

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1 Introduction

Productivity change is generically measured in index form as ratio of output quantity index over input quantity index. Specific measures materialize by selecting the output concept to be used (gross output or value added) and the number of inputs to be considered (resulting in single, multiple or total factor productivity indices); see Balk (2003). The two periods compared by such an index are usually years, since in the majority of cases the most natural accounting period appears to be a year.

Now there seems to be a demand from policy side for more frequent statistics, for example quarterly. This paper explores, from a theoretical perspective, the options for obtaining consistency between annual and quarterly (or more general: between period and subperiod) measures of productivity change.

The paper unfolds as follows. Section 2 considers the simple case of a single-input single-output production unit. Section 3 generalizes this to a multiple-input multiple-output situation, where input and output prices are fixed. Section 4 asks whether in a completely realistic case it is possible to obtain consistency between period and subperiod price, quantity and productivity indices. The conclusion, drawn in section 5, appears to be that such a complete consistency is impossible to obtain.

2 A simple case

Let us for a start consider a single-input single-output production unit through two adjacent periods, called 0 and 1 respectively, of equal length. Each period consists of M subperiods, also of equal length. The quantity of output produced during subperiod m of period t will be denoted by y^{tm} $(t = 0, 1; m = 1, ..., M)$. Likewise, the quantity of input used during subperiod m of period t will be denoted by x^{tm} $(t = 0, 1; m = 1, ..., M)$. All these quantities are assumed to be strictly positive.

The quantity of output produced during the entire period t is evidently measured by

$$y^t \equiv \sum_{m=1}^{M} y^{tm} \; (t = 0, 1). \tag{1}$$

Likewise, the quantity of input used during the entire period t is evidently measured by

$$x^t \equiv \sum_{m=1}^{M} x^{tm} \; (t = 0, 1). \tag{2}$$

In the case of a single-input single-output unit one can unambiguously talk about *productivity* as the quantity of output per unit of input. Hence, the productivity in subperiod m of period t is measured by

$$PROD(tm) \equiv y^{tm}/x^{tm} \; (t = 0, 1; m = 1, ..., M), \tag{3}$$

and the productivity in the entire period t by

$$PROD(t) \equiv y^t/x^t \; (t = 0, 1). \tag{4}$$

It is straightforward to check that the productivity of any period can be expressed as a weighted arithmetic average of its subperiod productivities,

$$PROD(t) = \sum_{m=1}^{M} (x^{tm}/x^t) PROD(tm), \tag{5}$$

the weights being input quantity shares. Alternatively, the productivity of any period can be expressed as a weighted harmonic average of its subperiod productivities,

$$PROD(t) = \left(\sum_{m=1}^{M} (y^{tm}/y^t)(PROD(tm))^{-1} \right)^{-1}, \tag{6}$$

the weights now being output quantity shares. A third possibility, less intuitive but useful at a later stage, is to employ the logarithmic mean[1] to obtain

$$\ln PROD(t) = \sum_{m=1}^{M} \frac{L(x^{tm}, y^{tm})}{L(x^t, y^t)} \ln PROD(tm). \tag{7}$$

[1]The logarithmic mean is, for two positive real numbers a and b, defined by $L(a,b) \equiv (a-b)/\ln(a/b)$ when $a \neq b$, and $L(a,a) \equiv a$. It has the following properties: (1) $\min(a,b) \leq L(a,b) \leq \max(a,b)$; (2) $L(a,b)$ is continuous; (3) $L(\lambda a, \lambda b) = \lambda L(a,b)$ $(\lambda > 0)$; (4) $L(a,b) = L(b,a)$; (5) $(ab)^{1/2} \leq L(a,b) \leq (a+b)/2$; (6) L(a,1) is concave.

Put otherwise, $PROD(t)$ is a weighted product of $PROD(tm)$ $(m = 1, ..., M)$, where the weights are symmetric in input and output quantities. Note however that, due to the concavity of $L(a, 1)$, the sum of these weights is less than or equal to 1. Hence, $PROD(t)$ is not a weighted geometric average of $PROD(tm)$ $(m = 1, ..., M)$.

Productivity change between two (sub-) periods, as measured in ratio form, is naturally defined as the ratio of the productivities of the two (sub-) periods considered. In this way the productivity change between periods 0 and 1 is measured by

$$IPROD(1, 0) \equiv \frac{PROD(1)}{PROD(0)} = \frac{y^1/x^1}{y^0/x^0}. \tag{8}$$

When considering subperiods, there are a number of possibilities. In line with the previous definition one could consider the productivity change between two adjacent subperiods $m - 1$ and m of period t; that is,

$$\begin{aligned} IPROD(tm, tm-1) &\equiv \frac{PROD(tm)}{PROD(t, m-1)} \\ &= \frac{y^{tm}/x^{tm}}{y^{t,m-1}/x^{t,m-1}} \; (t = 0, 1; m = 1, ..., M), \end{aligned} \tag{9}$$

where we will use the convention that subperiod 0 of period t is the same as subperiod M of period $t - 1$.

A second possibility is to compare the productivity of a certain subperiod to the productivity of the corresponding previous subperiod; that is,

$$IPROD(1m, 0m) \equiv \frac{PROD(1m)}{PROD(0m)} = \frac{y^{1m}/x^{1m}}{y^{0m}/x^{0m}} \; (m = 1, ..., M). \tag{10}$$

A third possibility is to compare the productivity of a certain subperiod to the productivity of the previous period; that is,

$$IPROD(1m, 0) \equiv \frac{PROD(1m)}{PROD(0)} = \frac{y^{1m}/x^{1m}}{y^0/x^0} \; (m = 1, ..., M). \tag{11}$$

These three are the most usual modes of comparison.

The interesting question now is: do there exist relations between subperiod productivity indices, of whatever type, and period indices? Let us first look at the subperiod-to-period type indices. Using expression (5), one simply checks that

$$IPROD(1,0) = \sum_{m=1}^{M} (x^{1m}/x^{1}) IPROD(1m,0); \tag{12}$$

that is, $IPROD(1,0)$ is a weighted mean of $IPROD(1m,0)$ $(m = 1,...,M)$. The weights are the subperiod input quantity shares of period 1, x^{1m}/x^{1} $(m = 1,...,M)$. If all these shares happen to be equal to $1/M$, then $IPROD(1,0)$ is an unweighted mean of $IPROD(1m,0)$ $(m = 1,...,M)$. Put otherwise, each subperiod index $IPROD(1m,0)$ can then be seen as an approximation to the period index $IPROD(1,0)$. The assumption, however, is rather strong and, moreover, concerns the comparison period 1, which is unfortunate from the viewpoint of computation in real time.

The relation between $IPROD(1,0)$ and the subperiod-to-corresponding-subperiod indices $IPROD(1m,0m)$ $(m = 1,...,M)$ is less simple. Again using expression (5) it appears that

$$IPROD(1,0) = \sum_{m=1}^{M} \frac{x^{1m} PROD(0m)}{x^{1} PROD(0)} IPROD(1m,0m); \tag{13}$$

that is, $IPROD(1,0)$ is a linear combination of $IPROD(1m,0m)$ $(m = 1,...,M)$. The weights $x^{1m}PROD(0m)/x^{1}PROD(0)$, however, don't add up to 1. Sufficient conditions for these weights to be equal to $1/M$ are that the subperiod input quantity shares are invariant through time, $x^{1m}/x^{1} = x^{0m}/x^{0}$ $(m = 1,...,M)$, and that all the output quantity shares of period 0 are the same, $y^{0m}/y^{0} = 1/M$ $(m = 1,...,M)$. From a practical point of view, these conditions are hard to justify.

Another attempt to link the period index $IPROD(1,0)$ and the subperiod indices $IPROD(1m,0m)$ $(m = 1,...,M)$ derives from a comparison of their definitions. Notice that $IPROD(1,0)$ can be expressed as

$$IPROD(1,0) = \frac{(1/M)\sum_{m=1}^{M} y^{1m} / (1/M)\sum_{m=1}^{M} x^{1m}}{(1/M)\sum_{m=1}^{M} y^{0m} / (1/M)\sum_{m=1}^{M} x^{0m}}. \tag{14}$$

Let δ^{tm} and ϵ^{tm} $(t = 0,1; m = 1,...,M)$ be defined by

$$\begin{aligned}\delta^{tm} &\equiv x^{tm} - x^t/M && (15)\\ \epsilon^{tm} &\equiv y^{tm} - y^t/M. && (16)\end{aligned}$$

A first-order Taylor series expansion then delivers

$$IPROD(1m, 0m) = IPROD(1, 0) + R(\delta^{0m}, \epsilon^{0m}, \delta^{1m}, \epsilon^{1m}), \qquad (17)$$

where the remainder term $R(.)$ tends to zero when its arguments tend to zero. Hence, if δ^{tm} and ϵ^{tm} randomly fluctuate around 0, then the subperiod-to-corresponding-subperiod indices can be seen as approximations to the period-to-period index. But again, the condition is hard to justify in practice, especially as far as period 1 is concerned.

The adjacent subperiod indices $IPROD(tm, tm-1)$ $(t = 0, 1; m = 1, ..., M)$ can be related to the subperiod-to-corresponding-subperiod indices by chaining,

$$IPROD(1m, 0m) = \qquad (18)$$
$$\prod_{\mu=1}^{m} IPROD(1\mu, 1\mu - 1) \prod_{\mu=m+1}^{12} IPROD(0\mu, 0\mu - 1) \; (m = 1, ..., M).$$

The right-hand side of (18) can then be inserted into expression (13) to obtain a relation between $IPROD(1, 0)$ and the adjacent subperiod indices. But this relation does not have a simple outlook.

The conclusion is that already in the extremely simple case of a single-input single-output unit it appears to be difficult to relate subperiod and period productivity indices to each other in a simple way.

3 A more realistic case

Let us now consider a production unit that produces L outputs and uses N inputs. The quantity of output ℓ produced during subperiod m of period t will be denoted by y_ℓ^{tm} $(\ell = 1, ..., L; t = 0, 1; m = 1, ..., M)$. Likewise, the quantity of input n used during subperiod m of period t will be denoted by x_n^{tm} $(n = 1, ..., N; t = 0, 1; m = 1, ..., M)$. All these quantities are assumed to be strictly positive.

The quantity of output ℓ produced during the entire period t is evidently measured by

$$y_\ell^t \equiv \sum_{m=1}^{M} y_\ell^{tm} \; (\ell = 1, ..., L; t = 0, 1). \tag{19}$$

Likewise, the quantity of input n used during the entire period t is evidently measured by

$$x_n^t \equiv \sum_{m=1}^{M} x_n^{tm} \; (n = 1, ..., N; t = 0, 1). \tag{20}$$

When there are multiple inputs and multiple outputs the concept of productivity (level) is no longer unambiguous. Prices are necessary to aggregate quantities. Thus, suppose we have output prices $p \equiv (p_1, ..., p_L)$ and input prices $w \equiv (w_1, ..., w_N)$. The aggregate output quantity produced during subperiod m of period t is then given by

$$p \cdot y^{tm} = \sum_{\ell=1}^{L} p_\ell y_\ell^{tm} \; (t = 0, 1; m = 1, ..., M), \tag{21}$$

where vector notation is used to simplify notation and highlight the analogies to the expressions in the previous section. One could also say that $p \cdot y^{ym}$ is the subperiod tm output value expressed in constant prices. The aggregate output quantity produced during the entire period t is naturally given by

$$p \cdot y^t = \sum_{\ell=1}^{L} p_\ell y_\ell^t = \sum_{m=1}^{M} p \cdot y^{tm} \; (t = 0, 1). \tag{22}$$

Likewise, the aggregate input quantity used during subperiod m of period t is given by

$$w \cdot x^{tm} = \sum_{n=1}^{N} w_n x_n^{tm} \; (t = 0, 1; m = 1, ..., M). \tag{23}$$

This is the subperiod tm input value expressed in constant prices. The aggregate input quantity used during the entire period t is also naturally given by

$$w \cdot x^t = \sum_{n=1}^{N} w_n x_n^t = \sum_{m=1}^{M} w \cdot x^{tm} \; (t = 0, 1). \tag{24}$$

Conditional on input prices w and output prices p, the productivity (level) in subperiod m of period t is measured by

$$PROD(tm) \equiv p \cdot y^{tm}/w \cdot x^{tm} \; (t = 0, 1; m = 1, ..., M), \tag{25}$$

and the productivity (level) in the entire period t by

$$PROD(t) \equiv p \cdot y^{t}/w \cdot x^{t} \; (t = 0, 1). \tag{26}$$

This can be expressed in terms of subperiod productivity levels in three ways, namely

$$PROD(t) = \sum_{m=1}^{M} (w \cdot x^{tm}/w \cdot x^{t}) PROD(tm), \tag{27}$$

$$PROD(t) = \left(\sum_{m=1}^{M} (p \cdot y^{tm}/p \cdot y^{t})(PROD(tm))^{-1} \right)^{-1}, \tag{28}$$

and

$$\ln PROD(t) = \sum_{m=1}^{M} \frac{L(w \cdot x^{tm}, p \cdot y^{tm})}{L(w \cdot x^{t}, p \cdot y^{t})} \ln PROD(tm). \tag{29}$$

The definitions of productivity change between two periods, between two subperiods, and between a subperiod and a period are straightforward. For instance, productivity change between periods 0 and 1 is measured by

$$IPROD(1, 0) \equiv \frac{PROD(1)}{PROD(0)} = \frac{p \cdot y^{1}/w \cdot x^{1}}{p \cdot y^{0}/w \cdot x^{0}}. \tag{30}$$

It is simple to check that the following relations hold:

$$IPROD(1, 0) = \sum_{m=1}^{M} (w \cdot x^{1m}/w \cdot x^{1}) IPROD(1m, 0), \tag{31}$$

and

$$IPROD(1, 0) = \sum_{m=1}^{M} \frac{w \cdot x^{1m}}{w \cdot x^{1}} \frac{PROD(0m)}{PROD(0)} IPROD(1m, 0m). \tag{32}$$

Moreover, analogous to the way it was done in the previous section, any subperiod-to-corresponding-subperiod productivity index $IPROD(1m, 0m)$ can be written as a chain of adjacent subperiod indices.

4 Is it possible to use general price and quantity indices?

It is clear that the productivity index $IPROD(1,0)$ can be re-expressed as

$$IPROD(1,0) = \frac{p \cdot y^1 / p \cdot y^0}{w \cdot x^1 / w \cdot x^0}; \tag{33}$$

that is, as the ratio of an output quantity index and an input quantity index. The same holds for the other productivity indices considered in the previous section.

These quantity indices have a specific functional form; they are so-called Lowe indices (see PPI Manual 2004). One of the drawbacks of these quantity indices is that the corresponding price indices violate a rather fundamental axiom. Consider for instance the output quantity index $p \cdot y^1 / p \cdot y^0$. The corresponding price index is obtained by dividing the quantity index into the value ratio $p^1 \cdot y^1 / p^0 \cdot y^0$, where p^t $(t = 0, 1)$ denotes the vector of period t output prices, the result being

$$\frac{p^1 \cdot y^1 / p \cdot y^1}{p^0 \cdot y^0 / p \cdot y^0}. \tag{34}$$

It is clear that this price index violates the identity axiom, which requires a price index to deliver the outcome 1 whenever the price vectors of the two periods compared are equal, $p^1 = p^0$. Such a violation is generally considered to be undesirable.

Thus it seems that one can do better by looking for more general functional forms $P(.), P'(.), Q(.), Q'(.)$, such that

$$p^1 \cdot y^1 / p^0 \cdot y^0 = P(p^1, y^1, p^0, y^0) Q(p^1, y^1, p^0, y^0) \tag{35}$$

$$w^1 \cdot x^1 / w^0 \cdot x^0 = P'(w^1, x^1, w^0, x^0) Q'(w^1, x^1, w^0, x^0), \tag{36}$$

and a reasonable number of fundamental axioms for price and quantity indices be satisfied. Here w^t $(t = 0, 1)$ denotes the vector of period t input prices. Notice that the functional forms used at the output side may or may not be the same as those used at the input side (apart from the dimension of the price and quantity vectors involved).

The productivity index for period 1 relative to period 0 could then be defined as

$$IPROD(1,0) \equiv \frac{Q(p^1, y^1, p^0, y^0)}{Q'(w^1, x^1, w^0, x^0)}; \tag{37}$$

that is, output quantity index divided by input quantity index.

Similarly, the productivity index for subperiod $1m$ relative to period 0 is defined as

$$IPROD(1m,0) \equiv \frac{Q(p^{1m}, y^{1m}, p^0, y^0)}{Q'(w^{1m}, x^{1m}, w^0, x^0)}, \tag{38}$$

where p^{tm} and w^{tm} ($t = 0, 1; m = 1, ..., M$) denote the vectors of subperiod output and input prices respectively. The relation between period and subperiod prices is given by

$$p^t_\ell \equiv \sum_{m=1}^{M} p^{tm}_\ell y^{tm}_\ell / y^t_\ell \; (\ell = 1, ..., L; t = 0, 1) \tag{39}$$

$$w^t_n \equiv \sum_{m=1}^{M} w^{tm}_n x^{tm}_n / x^t_n \; (n = 1, ..., N; t = 0, 1), \tag{40}$$

and the relation between period and subperiod output and input values is given by

$$p^t \cdot y^t = \sum_{m=1}^{M} p^{tm} \cdot y^{tm} \; (t = 0, 1) \tag{41}$$

$$w^t \cdot x^t = \sum_{m=1}^{M} w^{tm} \cdot x^{tm} \; (t = 0, 1). \tag{42}$$

Then, on the one hand the so-called profitability ratio can be decomposed as

$$\frac{p^1 \cdot y^1 / p^0 \cdot y^0}{w^1 \cdot x^1 / w^0 \cdot x^0} = \frac{P(p^1, y^1, p^0, y^0)}{P'(w^1, x^1, w^0, x^0)} \frac{Q(p^1, y^1, p^0, y^0)}{Q'(w^1, x^1, w^0, x^0)}. \tag{43}$$

But on the other hand, by temporal desaggregation, one obtains

$$\begin{aligned} \frac{p^1 \cdot y^1 / p^0 \cdot y^0}{w^1 \cdot x^1 / w^0 \cdot x^0} &= \sum_{m=1}^{M} \frac{w^{1m} \cdot x^{1m}}{w^1 \cdot x^1} \frac{p^{1m} \cdot y^{1m} / p^0 \cdot y^0}{w^{1m} \cdot x^{1m} / w^0 \cdot x^0} \\ &= \sum_{m=1}^{M} \frac{w^{1m} \cdot x^{1m}}{w^1 \cdot x^1} \frac{P(p^{1m}, y^{1m}, p^0, y^0)}{P'(w^{1m}, x^{1m}, w^0, x^0)} \frac{Q(p^{1m}, y^{1m}, p^0, y^0)}{Q'(w^{1m}, x^{1m}, w^0, x^0)}. \end{aligned} \tag{44}$$

By combining the previous two expressions, using the definitions (37) and (38), one obtains

$$IPROD(1,0) = \tag{45}$$
$$\sum_{m=1}^{M} \frac{w^{1m} \cdot x^{1m}}{w^1 \cdot x^1} \frac{P(p^{1m}, y^{1m}, p^0, y^0)/P(p^1, y^1, p^0, y^0)}{P'(w^{1m}, x^{1m}, w^0, x^0)/P'(w^1, x^1, w^0, x^0)} IPROD(1m, 0).$$

This is not a particularly simple relationship between period-to-period and subperiod-to-period productivity indices. Moreover, since the productivity index at the left-hand side is based on the same functional form(s) for the quantity indices as the productivity indices at the right-hand side, this relation imposes restrictions on those functional forms. It turns out that these restrictions are impossible to satisfy, except when $Q(.)$ and $Q'(.)$ exhibit the Lowe functional form. But then $P(.)$ and $P'(.)$ violate the fundamental identity axiom.

5 Conclusion

It appears that the high goal of full consistency between period and subperiod price, quantity and productivity indices is unattainable. But that means that choices must be made.

The first choice concerns what is to be seen as the most natural accounting period for the production unit. In most, if not all, cases this will be a year. Annual price, quantity, and productivity comparisons can be based on indices that satisfy the basic axioms and together form a consistent system.

Given the need for subannual productivity information, the second choice concerns the type of index to use. As shown, every choice entails at best an approximate relationship between subannual and annual indices. The nature of this approximation should be clearly communicated to the public.

References

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